On the Ternary Quadratic Diophantine Equation $5(x^2 + y^2) - 6xy = 4z^2$

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Abstract
The ternary quadratic Diophantine equation $5(x^2 + y^2) - 6xy = 4z^2$ is analyzed for its non-zero distinct integer solutions. Five different patterns of integral solutions are obtained. Few interesting relations among the solutions and some special polygonal numbers are presented.

Keywords: Ternary quadratic equation, Integral solutions.

Mathematical Classification: 11D09.

1. Introduction:
Ternary quadratic equations are rich in variety. For more detailed understanding one can see [1-7]. For the non-trivial integral solutions of ternary quadratic Diophantine equations [8-9] has been studied. [10-12] has been referred for various ternary quadratic Diophantine equations.

In this communication, we consider yet another interesting ternary quadratic equation $5(x^2 + y^2) - 6xy = 4z^2$ and obtain different patterns of non-trivial integral solutions. Also, a few interesting relations among the solutions and special polygonal, Centered, Gnomonic, Star numbers are presented.

Notations:

- $T_{m,n}$ = Triangular Number of rank $n$.
- $Gno_n$ = Gnomonic Number of rank $n$.
- $P(n)$ = Pronic Number of rank $n$.
- $Star_n$ = Star Number of rank $n$.
- $CH_n$ = Centered Hexagonal Number of rank $n$.

METHOD OF ANALYSIS
Consider the equation $5(x^2 + y^2) - 6xy = 4z^2$ ... (1)
The substitution of the linear transformation $x = u + v; y = u - v$ ... (2)
in (1) gives $u^2 + 4v^2 = z^2$ ... (3)

We present below different methods of solving (3) and thus obtain different choices of integer solutions of (1)

Pattern-1:
The general solution of (3) is
$u = r^2 - s^2$;
$v = rs$;
$z = r^2 + s^2$

Employing the values of $u$ and $v$ in (2) we get the non-zero distinct solution of (1) as
$x = r^2 - s^2 + rs$
y = $r^2 - s^2 - rs$
z = $r^2 + s^2$

Observations:
1. $1), 33 = - + r x Gno CH r r$
2. $) 4 \mod 0 ) 1, ( r y CH r r$
3. $0) 1, 2, ( 2, 3 = - + r x TCH n r$
4. $0) 1, 2, ( - + - r r r Gno Trzryrx$
5. $) 8 \mod 0 7, 1, 3, 1, 3, 1, 3, 20 \equiv - - - + - r Gno Trzry$.
6. , , , , , , , , and are perfect squares

Pattern-2:
Let $z = a^2 + 4b^2$ ...(4)

Using (4) in (3) and applying the method of factorization, define

$u + i2v)(u - i2v) = [(a + i2b)(a - i2b)]^2$

Equating real and imaginary parts the value of $u$ and $v$ are obtained as
$u = a^2 - 4b^2; v = 2ab$; ...(5)

From (4) and (5) the non-zero distinct solution of (1) is

- $x = a^2 - 4b^2 + 2ab$
- $y = a^2 - 4b^2 - 2ab$
- $z = a^2 + 4b^2$

Observations:
1. $z(2a, a) = x(2a, a)$ and $y(2a, a) + z(2a, a)$ are perfect squares.
2. $2[x(a,1) + y(a,1)]$ can be expressed as the difference of 2 square numbers.
3. $x(2a,1) + y(2a,1) + z(2a,1) \equiv 0 \mod 4$
4. The expressions $x(4a, a) + y(4a, a)$ and
z(4a, a) + y(4a, a) represents nasty number
5. \( x(4a, l) - y(4a, l) + z(4a, l) - 32T_{a, l} = 0 \) (mod 4)
6. \( x(3a, a) - y(3a, a) + z(3a, a) - T_{3a, a} - 14Gno_x = 0 \) (mod 50)

Pattern-3:

(3) can be written as \( u^2 + 4v^2 = z^2 \) \( \ldots (6) \)

Write \( 1 = \frac{(3 + 4i)(3 - 4i)}{25} \) \( \ldots (7) \)

Using (4) and (6) in (3) and proceeding as in pattern2 we get the non-zero distinct integer solutions of (1) as
\[
\begin{align*}
x &= 25A^2 - 100B^2 - 50AB \\
y &= 5A^2 - 20B^2 - 110AB \\
z &= 25A^2 + 100B^2
\end{align*}
\]

Observations:
1. \( y(A, l) + z(A, l) - 10CH_{a, l} + 4Gno_{A, l} = 0 \) (mod 30)
2. \( 100CH_{a, l} - 3x(2, A, l) = 0 \) (mod 200)
3. \( 20T_{2a, a} - 60Gno_x - x(2, A, l) + y(2, A, l) + z(2, A, l) = 0 \) (mod 80)
4. \( y(A, 1) + z(A, 1) - 10\{Star_{a} + T_{a, a} - 2Gno_x\} = 0 \)
5. \( z(A, A) - [x(A, A)] \) and \( z(B, 2B) - x(B, 2B) \) are perfect squares

Pattern-4:

Write \( 1 = \frac{(6 + 8i)(6 - 8i)}{100} \) \( \ldots (8) \)

Using (4) and (8) in (6) and proceeding as in pattern3 we get the non-zero distinct solutions of (1) to be
\[
\begin{align*}
x &= 100A^2 - 400B^2 - 200AB \\
y &= 20A^2 - 80B^2 - 440AB \\
z &= 100A^2 + 400B^2
\end{align*}
\]

Observations:
1. \( [x(A, A), y(A, A)] \) is a quadratic integer
2. \( 200CH_{a, l} + 100Gno_{A, l} - 200T_{a, l} - [x(A, 1) + z(A, 1)] = 0 \) (mod 100)
3. \( y(A, 1) + z(A, 1) + 160Gno_x - 20Star_{a} = 0 \) (mod 140)
4. \( 400\{Pr(A) - Gno_{x}\} - [x(2, A, l) + z(2, A, l)] = 0 \)
5. \( y(2, A, l) + z(2, A, l) + 200Gno_{a, l} - 80Star_{a} = 0 \) (mod 40)

Pattern-5:

Write \( 1 = \frac{(11 + 60i)(11 - 60i)}{61^2} \) \( \ldots (9) \)

Using (4) and (8) in (6) and proceeding as in pattern4 we get the non-zero distinct solutions of (1) to be
\[
\begin{align*}
x &= 2501A^2 - 10004B^2 - 13298AB \\
y &= -1159A^2 + 4636B^2 - 15982AB \\
z &= 3721A^2 + 14884B^2
\end{align*}
\]

Observations:
1. \( [T_{a, a} + 332T_{a, l} + 715Gno_{x, a}] - [x(A, l) + y(A, l)] = 0 \) (mod 19670)
2. \( x(A, l) + z(A, l) + 4316Gno_{a, l} - 15557T_{a, a} - T_{a, l} = 0 \) (mod 564)
3. \( 3.3T_{a, l} + 1240T_{a, a} - 6020Gno_{a, l} - x(A, l) = 0 \) (mod 16024)
4. \( -[T_{3a, a} + 1150Pr(A) + 7375Gno_{x} + y(A, l)] = 0 \) (mod 2739)
5. \( 5.286T_{3a, l} + 1717Gno_{a, l} + T_{3a, a} - z(A, l) = 0 \) (mod 16601)

CONCLUSION:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by \( 5(x^2 + y^2) - 6xy = 4z^2 \)
One can also search for other patterns of solutions for the equation.

REFERENCES:
[12].Gopalan MA, Vidhyalakshmi S, Mallika S. On ternary quadratic Diophantine equation \( 8(x^2 + y^2) - 15xy = 80z^2 \) BOMSR, 2014; 2(4):429-433