Degree Of Approximation Of Function \( f \in \text{Lip} (\psi(t), p) \) Class By \( K^\lambda \) - Summability Means

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**Abstract:** In the present paper, we study the degree of approximation of function \( f \in \text{Lip} (\psi(t), p) \) by \( K^\lambda \) - summability means of its Fourier series and conjugate of function \( f \in \text{Lip} (\psi(t), p) \) by \( K^\lambda \) - summability means of its conjugate Fourier series.

**Keywords:** Degree of approximation, Lip (\( \psi(t), p) \) class of functions, Fourier series, conjugate Fourier series, summability means of its conjugate Fourier series.

**1. Introduction:**

The method \( K^\lambda \) was first introduced by Karatama[2] and Lotosky[4] reintroduced the special case \( \lambda = 1 \). Only after the study of Agnew[1] similar cases. Working in the same direction Ojha[3], Tripathi and Lal[7] have studied \( K^\lambda \)-summability of Fourier series under different conditions.

Nigam and Sharma[6] obtained the degree of approximation by Karamatassummability method. In the present paper, we have obtained new theorems on degree of approximation of function \( f \) belonging to \( \text{Lip} (\psi(t), p) \) by \( K^\lambda \) - means on its Fourier series and conjugate Fourier series.

**2. Definition and notations:**

Let us define, for \( n = 0,1,2,\ldots \), the numbers\( \{\frac{n}{m}\} \) for \( 0 \leq m \leq n \) by

\[
\prod_{n=0}^{n-1}(x + v) = \sum_{m=0}^{n}\frac{n!}{m!} x^m (\frac{x+n}{\Gamma(x)} = x(x+1)(x+2)\ldots(x+n-1)
\]

(2.1)

The numbers\( \{\frac{n}{m}\} \) are known as the absolute value of stirling number of first kind.

Let \( \{s_n\} \) be the sequence of partial sums of an infinite series \( \sum u_n \), and let us write

\[
s_n^\lambda = \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{m=0}^{n}\frac{n!}{m!} \lambda^m s_m
\]

(2.2)

To denote the \( n^{th} \) \( K^\lambda \) mean of order \( \lambda > 0 \). If \( s_n^\lambda \rightarrow s \) as \( n \rightarrow \infty \), where \( s \) is a fixed finite number, then the sequence \( \{s_n\} \) or the series \( \sum u_n \) is said to be summable by Karamata method (\( K^\lambda \)) of order \( \lambda > 0 \) to the sum \( s \) and we can write

\[
s_n^\lambda \rightarrow s(\text{K}^\lambda) \text{as} \ n \rightarrow \infty.
\]

(2.3)

Let \( f \) be a periodic function with period \( 2\pi \) and integrable in the Lebesgue sense. The Fourier series be given by

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = I(x), \quad 0 < x < \pi.
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The conjugate series of Fourier series (2.4) is given by

\[
\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x)
\]

(2.5)with \( n^{th} \) partial sums \( s_n(f; x) \).

A \( 2\pi \) - periodic function \( f(x) \) is said to belong to the class \( \text{Lip} (\psi(t), p), \) \( p > 1 \) if

\[
|f(x) - f(x + t)| \leq M(\psi(t) t^{-1/p}), \quad 0 < t < \pi.
\]

(2.6)

where \( \psi(t) \) is a positive increasing function and \( M \) is a positive number independent of \( x \) and \( t \).

**L\(_x\)**-norm of a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is defined by

\[
||f||_x = \sup \{|f(x)|: x \in \mathbb{R}\}
\]

(2.7)

**L\(_r\)**-norm is defined by

\[
||f||_r = \left( \int_0^{2\pi} |f(x)|^r dx \right)^{1/r}, \quad r \geq 1
\]

(2.8)

The degree of approximation of a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) by a trigonometric polynomial \( t_n \) of degree \( n \) under super norm \( ||f||_s \) is defined by

\[
\text{degree of approximation of } f \text{ by } t_n \text{ under super norm } ||f||_s
\]

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\[ \|t_n - f\|_\alpha = \sup \{|t_n(x) - f(x)| : x \in \mathbb{R}\} \]

(2.9)

A function \( f \in \text{Lip}_\alpha \) if
\[ |f(x + t) - f(x)| = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1 \]

(2.10)

And \( f \in \text{Lip}(a, p) \) if
\[ \left\{ \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right\}^{\frac{1}{p}} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, \quad p \geq 1 \]

(2.11)

Given a positive increasing function \( \psi(t) \) and an integer \( p \geq 1 \), then \( f \in \text{Lip}(\psi(t), p) \) if
\[ \left\{ \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right\}^{\frac{1}{p}} = O(\psi(t)) \]

(2.12)

We shall use following notation:
\[ \phi(t) = f(x + t) + f(x - t) - 2f(x)K_n(t) = \sum_{m=\text{odd}}^{n} \frac{|\lambda|^m \sin \left( \frac{m+1}{2} t \right)}{\Gamma(\lambda + m) \sin \left( \frac{t}{2} \right)} \psi(t) = f(x + t) + f(x - t) \]

\[ f(x - t)K_n(t) = \sum_{m=\text{odd}}^{n} \frac{|\lambda|^m \cos \left( \frac{m+1}{2} t \right)}{\Gamma(\lambda + m) \cos \left( \frac{t}{2} \right)} f(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \left( \frac{t}{2} \right) \, dt \]

3. The main results:

3.1 Theorem 1: Let \( f \) is a \( 2\pi \)-periodic function, Lebesgue integrable on \([-\pi, \pi]\) and \( f \in \text{Lip}(\psi(t), p) \) then its degree of approximation by \( K^\lambda \)-summability means on its Fourier series is given by
\[ ||s_n - f||_r = O \left\{ \frac{\lambda \log(n + 1)}{n + 1} \left( \frac{1}{n + 1} \right) \left( \frac{1}{n + 1} \right) \left( \frac{1}{n + 1} \right) \right\} \]

(3.1)

where \( s_n \) is \( K^\lambda \)-means of Fourier series (2.4).

We shall use following conditions (see Khan[3])
\[ \left\{ \frac{1}{n+1} \left( \frac{\psi(t)}{t^r} \right)^p \right\}^{\frac{1}{r}} = O \left( \frac{1}{n + 1} \right) \]

(3.2)

And
\[ \left\{ \int_0^\pi \left( \frac{\psi(t)}{t^{r+1}} \right)^q \, dt \right\}^{\frac{1}{r}} = O \left( (n + 1)^2 \psi \left( \frac{1}{n + 1} \right) \right) \]

(3.3)

where \( \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq r \leq \infty \)

3.2 Theorem 2: If a function \( f \), conjugate to a \( 2\pi \)-periodic function \( \hat{f} \in \text{Lip}(\psi(t), p) \) then its degree of approximation by \( K^\lambda \)-summability means on its conjugate Fourier series is given by
\[ ||\hat{s}_n - \hat{f}||_r = \left. O \left( \frac{\lambda \log(n + 1)}{n + 1} \right) \left( \frac{1}{n + 1} \right) \left( \frac{1}{n + 1} \right) \right\} \]

(3.4)

where \( \hat{s}_n \) is \( K^\lambda \)-means of conjugate Fourier series (2.5).

4. Lemma:

For the proof of our theorem following lemmas are required. For the proof of lemma see Nigam and Sharma [17].

4.1 Lemma 1: Let \( \lambda > 0 \) and \( 0 < t < \frac{\pi}{2} \) then
\[ \frac{\text{Im} \Gamma(\lambda e^{it} + n)}{\Gamma(\lambda \cos t + n) \sin \left( \frac{\pi}{2} \right)} = | \sin(\lambda \log(n + 1) \cdot \sin t) | \frac{1}{\sin \left( \frac{\pi}{2} \right)} \]

+ \( O(1) \)
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \phi(m) = \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(m + \frac{1}{2})t}{\sin\frac{t}{2}} dt \]

The proof of the theorem involves the use of partial sums and the evaluation of certain integrals. The partial sum \( S_n(x) \) of a series is defined as:

\[ S_n(x) = \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin(m + \frac{1}{2})t}{\sin\frac{t}{2}} dt \]

Using this definition and properties of the \( \phi(t) \) function, we can show that the partial sum converges to the desired limit. The proof involves the application of integral calculus and properties of trigonometric functions.

The final result shows that the limit of the partial sum as \( n \to \infty \) gives the desired asymptotic behavior.

\[ \lim_{n \to \infty} S_n(x) = 0 \]

This completes the proof of the theorem 1.
\[ I_3 = \int_0^1 \psi(t) \bar{R}_n(t) \, dt \]

By lemma 3

\[ I_3 = O \left( \int_0^1 e^{-\frac{1}{2} \log(n+1)} \sin(n \log(n) \cdot \sin t) \, dt \right) \]

\[ + O \left( \frac{1}{n} \int_0^1 \left| \psi(t) \right| \, dt \right) \]

\[ = I_{3.1} + I_{3.2} + I_{3.3} \]

(5.5)

Now

\[ I_{3.1} = O \left( \int_0^1 e^{-\frac{1}{2} \log(n+1)} \frac{1}{n} \left| \psi(t) \right| \, dt \right) \]

\[ = \left\{ \int_0^1 \left( \psi(t) \right)^p \, dt \right\} \left\{ \int_0^1 e^{-\frac{1}{2} \log(n+1)} \, dt \right\} \frac{1}{n} \]

\[ = O \left( \frac{1}{n} \right) \]

(5.6)

Next

\[ I_{3.2} = O \left( \log(n+1) \right) \left\{ \int_0^1 \left| \sin(n \log(n) + 1) \cdot \sin t) \right| \left| \psi(t) \right| \, dt \right\} \]

Since, for \( 0 < t < \frac{1}{n+1}, \sin nt \leq nt \)

\[ I_{3.2} = O \left( \log(n+1) \right) \left\{ \int_0^1 t \left| \psi(t) \right| \, dt \right\} \]

\[ = O \left( \log(n+1) \right) \left\{ \int_0^1 \left( \psi(t) \right)^p \, dt \right\} \left\{ \int_0^1 (t)^q \, dt \right\} \frac{1}{n} \]

\[ = O \left( \frac{1}{n} \right) \]

(5.7)

Now

\[ I_{3.3} = O \left( \int_0^1 e^{-\frac{1}{2} \log(n+1)} \sin \left( \frac{t}{n+1} \right) \left| \psi(t) \right| \, dt \right) \]

\[ = O \left( \int_0^1 \left( \frac{1}{n+1} \right) \left( \frac{1}{n+1} \right) \frac{1}{(n+1)^{q+1}} \right) \]

(5.8)

Combining (5.5), (5.6), (5.7) and (5.8)

\[ I_3 = O \left( \frac{1}{n+1} \right) \left\{ \frac{1}{(n+1)^{-1/2}} \right\} \]

\[ + O \left( \log n \right) \left( 1 + \frac{1}{n} \right) \]

(5.9)
\[ s_m(x) - f(x) = O \left( \frac{1}{n+1} \right) \left( \frac{1}{(n+1)^5} \right) \left( \frac{1}{(n+1)^7} \right) \]

Combining (5.4), (5.9) and (5.10)

\[ s_m(x) - f(x) = O \left( \frac{1}{n+1} \right) \left( \frac{1}{(n+1)^5} \right) \left( \frac{1}{(n+1)^7} \right) \]

This completes the proof of the theorem 2.

6. References: