On Generalized $BR$ – Recurrent Affinely Connected Space

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Abstract. In this paper, we introduced the generalized $BR$ – recurrent Finsler space, i.e. characterized by the following condition

$$B_m R_{jk}^i = \lambda_m R_{jk}^i + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh}) , \quad R_{jk}^i \neq 0 ,$$

which is an affinely connected space, we called it a generalized $BR$ – recurrent affinely connected space, where $B_m$ is Berwald’s covariant differential operator with respect to $x^m$, $\lambda_m$ and $\mu_m$ are known as recurrence vectors.

The purpose of the present paper to develop the above space by study the recurrence vectors field and to obtain the condition for some tensors to be recurrent in the generalized $BR$ – recurrent affinely connected space. Also to obtain different theorems for some tensors satisfy in above space. Various identities are established in our space.

Keywords: Generalized $BR$ – recurrent affinely connected space, Generalized $B$ – recurrent of Berwald curvature tensor and Generalized $B$ – recurrent of Weyl’s curvature tensor.

1.Introduction


Let $F$ be an n-dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [11]. The vector $y_i$ is defined by

$$y_i = g_{ij}(x, y)y^j .$$

The two sets of quantities $g_{ij}$ and its associative $g^{ij}$, which are components of a metric tensor connected by

$$(1.1) \quad g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 \text{ if } j = k , \\ 0 \text{ if } j \neq k . \end{cases}$$

In view of (1.1) and (1.2), we have

$$(1.2) \quad g_{ij}g^{ik} = \delta^k_i = \begin{cases} 1 \text{ if } j = k , \\ 0 \text{ if } j \neq k . \end{cases}$$

In view of (1.1) and (1.2), we have

$$(1.3) \quad \delta^i_j y_i = y_k ,$$

$$(1.4) \delta^i_j y^k = y^j$$

and

$$(1.5) \quad \delta^i_j g_{ir} = g_{jr}.$$ The tensor $C_{ijk}$ is defined by

$$C_{ijk} = \frac{1}{2} \delta_{ik} g_{lj}$$

which is positively homogeneous of degree -1 in $y^i$ and symmetric in all its indices and called $(h)h$-torsion tensor [13] and its associative $C^i_{jk}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in its lower indices and called $(v)h$-torsion tensor. According to Euler’s theorem on homogeneous functions, these tensors satisfy the following:

$$(1.6) \quad C_{ijk} y^i = C_{ki}y^i = 0 \quad \text{and}$$

$$(1.7) \quad C^i_{jk} y^k = C^i_{kj} y^k.$$ The unit vector $l^i$ in the direction of $y^i$ is given by

$$l^i := \frac{y^i}{f}.$$ Berwald’s covariant derivative $B_k T^l_j$ of an arbitrary tensor field $T^l_j$ with respect to $x^m$ is given by

$$B_k T^l_j := \partial_k T^l_j - (\delta^l_j T^r_j) G^r_k + T^r_l G^l_j.$$ The processes of Berwald’s covariant differentiation and the partial differentiation, for an arbitrary tensor field $T^l_j$, commute according to

$$\partial_k (B_h B_k \partial_h) T^l_j = T^r_l G^l_j - T^l_j G^l_k.$$ Berwald’s covariant derivative of the vector $y^i$ vanish identically, i.e.

$$(1.9) \quad B_k y^k = 0 .$$

The $h(v)$- torsion tensor satisfies the relation

$$(1.10) \quad H^l_{kh} y^k = H^l_k = -H^l_{kh} y^k .$$ Berwald curvature tensor $H^l_{kh}$ and the $h(v)$ – torsion tensor $H^l_{kh}$ are skew-symmetric in the
lower indices $k$ and $h$ and they are positively homogenous of degree zero and one in $y^i$, respectively. They are satisfy the following:

(1.11) $H^i_{jk,h}y^i = H^i_{kh}$

and

(1.12) $H^i_{ki} = H^i_k$.

The deviation tensor $H^i_k$ is positively homogeneous of degree two in $y^i$. In view of Euler’s theorem on homogeneous functions we have the following relations

(1.13) $H = \frac{1}{n-1}H^i_i$,

where $H$ is the curvature scalar.

The tensor $H^i_{jk,h}$ defined by

(1.14) $H^i_{jk,h} = g_{ik}H^i_{jh}$.

2. Generalized $BR$–Recurrent Affinely Connected Space

A Finsler space whose connection parameter $G^i_{kh}$ is independent of $y^i$ is called an affinely connected or Berwald’s space. Thus, an affinely connected or Berwald’s space is characterized by any one of the equivalent conditions

(2.1) a) $G^i_{jk,h} = 0$

and

b) $C^i_{ijk,h} = 0$.

The connection parameters $\Gamma^i_{jk,h}$ of Cartan and $G^i_{kh}$ of Berwald coincide in affinely connected space and there are independent of directional argument [11], i.e. the conditions

(2.2) a) $\delta \Gamma^i_{jk,h} = 0$

and

b) $\delta \Gamma^i_{jk,h} = 0$.

In affinely connected space, Berwald’s covariant derivative of the metric tensor $g_{ij}$ vanishes, i.e.

(2.3) $\mathcal{B}_m g_{ij} = 0$.

The generalized $BR$–recurrent Finsler space which is characterized by the condition [4 - 6]

(2.4) $\mathcal{B}_m R^i_{jk,h} = \lambda_m R^i_{jk,h} + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

which briefly denoted by $G(BR) - RF_m$, where $\lambda_m$ is Berwald’s covariant differential operator with respect to $x^m$, $\lambda_m$ and $\mu_m$ are known as recurrence vectors and the tensor which satisfies the conditions (2.4) is called generalized $\mathcal{B}$–recurrent and briefly denoted by $GB - R$.

In this paper we shall introduce definition for $G(BR) - RF$, possess the properties of affinely connected space as follows:

**Definition 2.1.** A Finsler space whose Cartan’s third curvature tensor $R^i_{jk,h}$ satisfies the condition (2.4) which is an affinely connected space [satisfies the conditions (2.1a), (2.1b), (2.2a) and (2.2b)] will be called a generalized $BR$–recurrent affinely connected space and we shall denote it briefly by $G(BR) - R - \text{affinely connected space}$.

**Remark 2.1.** It will be sufficient to call the tensor which satisfies the condition of $G(BR) - R$–affinely connected space as a generalized $\mathcal{B}$–birecurrent tensor (briefly $GB - R$).

Let us consider $\alpha G(BR) - R - \text{affinely connected space}$.

The equations (2.2) [6] and (2.10) [4]

$\mathcal{B}_m R^i_{jk,h} = \lambda_m R^i_{jk,h} + \mu_m (g_{ij} g_{kh} - g_{ki} g_{jh}) + 2R^i_{jk,h} g^h \mathcal{B}_h C_{ilm}$

and

$\mathcal{B}_m W^i_{jk,h} = \lambda_m W^i_{jk,h} + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

+ $\frac{2}{n-1} R^i_{jk,h} g^h \mathcal{B}_h C_{jkm}$

and the equation (4) p.113

$\mathcal{B}_m P^i_{jk,h} = \lambda_m P^i_{jk,h} + (g_{jr} g_{kh} - g_{kr} g_{jh})$

- $2 P^i_{jk,h} g^h \mathcal{B}_h C_{irm}$

become

(2.5) $\mathcal{B}_m R^i_{jk,h} = \lambda_m R^i_{jk,h} + \mu_m (g_{ij} g_{kh} - g_{ki} g_{jh})$

(2.6) $\mathcal{B}_m W^i_{jk,h} = \lambda_m W^i_{jk,h} + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

and

(2.7) $\mathcal{B}_m P^i_{jk,h} = \lambda_m P^i_{jk,h} + \mu_m (g_{jr} g_{kh} - g_{kr} g_{jh})$

respectively.

Thus, we conclude

**Theorem 2.1.** In $G(BR) - R – \text{affinely connected space}$, the associate curvature tensor $R^i_{jk,h}$ of Cartan’s third curvature tensor $R^i_{jk,h}$, Weyl’s projective curvature tensor $W^i_{jk,h}$ and the associate curvature tensor $P^i_{jk,h}$ of Cartan’s second curvature tensor $P^i_{jk,h}$ are $GB - R$.

Now, if $\delta \lambda_m = 0$ and $\delta \mu_m = 0$, then the equation (2.2) [5]

$\mathcal{B}_m H^i_{jk,h} + H^i_{kh} G^i_{mr} - H^i_{rk} G^i_{mr} - H^i_{mr} G^i_{rk} = (\delta \lambda_m) H^i_{kh} + (\lambda_m H^i_{kh})$

$+ (\delta \mu_m) (g_{ir} g_{kh} - g_{kr} g_{jh}) + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

+ $2 y^i \mu_m C^i_{kh}$

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**Definition 2.1.** A Finsler space whose Cartan’s third curvature tensor $R^i_{jk,h}$ satisfies the condition (2.4) which is an affinely connected space [satisfies the conditions (2.1a), (2.1b), (2.2a) and (2.2b)] will be called a generalized $BR$–recurrent affinely connected space and we shall denote it briefly by $G(BR) - R - \text{affinely connected space}$.

**Remark 2.1.** It will be sufficient to call the tensor which satisfies the condition of $G(BR) - R – \text{affinely connected space}$ as a generalized $\mathcal{B}$–birecurrent tensor (briefly $GB - R$).

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and

$\mathcal{B}_m W^i_{jk,h} = \lambda_m W^i_{jk,h} + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

+ $\frac{2}{n-1} R^i_{jk,h} g^h \mathcal{B}_h C_{jkm}$

and the equation (4) p.113

$\mathcal{B}_m P^i_{jk,h} = \lambda_m P^i_{jk,h} + (g_{jr} g_{kh} - g_{kr} g_{jh})$

- $2 P^i_{jk,h} g^h \mathcal{B}_h C_{irm}$

become

(2.5) $\mathcal{B}_m R^i_{jk,h} = \lambda_m R^i_{jk,h} + \mu_m (g_{ij} g_{kh} - g_{ki} g_{jh})$

(2.6) $\mathcal{B}_m W^i_{jk,h} = \lambda_m W^i_{jk,h} + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

and

(2.7) $\mathcal{B}_m P^i_{jk,h} = \lambda_m P^i_{jk,h} + \mu_m (g_{jr} g_{kh} - g_{kr} g_{jh})$

respectively.

Thus, we conclude

**Theorem 2.1.** In $G(BR) - R – \text{affinely connected space}$, the associate curvature tensor $R^i_{jk,h}$ of Cartan’s third curvature tensor $R^i_{jk,h}$, Weyl’s projective curvature tensor $W^i_{jk,h}$ and the associate curvature tensor $P^i_{jk,h}$ of Cartan’s second curvature tensor $P^i_{jk,h}$ are $GB - R$.

Now, if $\delta \lambda_m = 0$ and $\delta \mu_m = 0$, then the equation (2.2) [5]

$\mathcal{B}_m H^i_{jk,h} + H^i_{kh} G^i_{mr} - H^i_{rk} G^i_{mr} - H^i_{mr} G^i_{rk} = (\delta \lambda_m) H^i_{kh} + (\lambda_m H^i_{kh})$

$+ (\delta \mu_m) (g_{ir} g_{kh} - g_{kr} g_{jh}) + \mu_m (\delta^i_j g_{kh} - \delta^i_k g_{jh})$

+ $2 y^i \mu_m C^i_{kh}$

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In this paper we shall introduce definition for $G(BR) - RF$, possess the properties of affinely connected space as follows:

**Definition 2.1.** A Finsler space whose Cartan’s third curvature tensor $R^i_{jk,h}$ satisfies the condition (2.4) which is an affinely connected space [satisfies the conditions (2.1a), (2.1b), (2.2a) and (2.2b)] will be called a generalized $BR$–recurrent affinely connected space and we shall denote it briefly by $G(BR) - R - \text{affinely connected space}$.
In Theorem 2.4, we conclude that if and only if (2.11) is satisfied, the identity (1.9) holds. Since $\lambda_m \neq 0$, the equations (5) become

\[ \mathcal{B}_m H_{hk} = \lambda_m H_{hk} + \mu_m g_{kh} - \delta_k y_h \]

respectively. Hence, we conclude Theorem 2.5: In $G(\mathcal{B}) - R$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the $h$-Ricci tensor $H_{hk}$ behaves as a recurrent.

We know that the Berwald curvature tensor satisfies the identity [11]

\[ B_m H_{jk}^h + B_k H_{jm}^h + B_k H_{jm}^h + H_{kl} G_{ijkl}^h + H_{kl}^h G_{ijkl}^h + H_{km} G_{ijkl}^h = 0. \]

Let us consider $G(\mathcal{B}) - R$ affinely connected space.

Suppose $\delta \lambda_m, \delta \mu_m = 0$ and by using (22) [5] in (2.16), we get

\[ \lambda_m H_{jk}^h + \lambda_k H_{mk}^h + \lambda_k H_{kh}^h + \mu_m (\delta^k y_h - \delta_k y_h) + \mu_k (\delta^k y_m - \delta_k y_m) + 2y^m \mu_m C_{jk}^m + 2y^m \mu_k C_{km}^m = 0. \]

Transvecting (2.17) by $y^p$, using (1.9), (1.11), (1.4), (1.1) and (1.6), we get

\[ \lambda_m H_{jk}^h + \lambda_k H_{mk}^h + \lambda_k H_{kh}^h + \mu_m (\delta^k y_h - \delta_k y_h) + \mu_k (\delta^k y_m - \delta_k y_m) + 2y^m \mu_m C_{jk}^m = 0. \]

Thus, we conclude Theorem 2.6: In $G(\mathcal{B}) - R$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the $h$-v torsion tensor $H_{hk}$ is $GB - R$.

Transvecting (2.19) by $y^k$, using (1.9), (1.10), (1.1) and (1.4), we get

\[ B_m H_{jk}^h = \lambda_m H_{jk}^h + \mu_m (\delta^k y_h - \delta_k y_h) \]

Thus, we conclude Theorem 2.7: In $G(\mathcal{B}) - R$ affinely connected space, if $\delta \lambda_m = 0, \delta \mu_m = 0$, then the deviation tensor $H_{jk}^h$, the curvature vector $H_k$, and the

\[ H_{kr,s} G_{m,jh} + \left( \delta \lambda_m \right)_m H_{kr,h} - \left( \lambda_m \right)_m H_{kr,h} \]

becomes

\[ (2.10) \quad \mathcal{B}_m H_{jk}^h = \lambda_m \mu_m g_{kh} - \left( \eta_m \right)_m H_{kr,h} + \left. \mu_m \right)_m H_{kr,h} - 2y^m \mu_m C_{ijh}. \]

This shows that

\[ (2.11) \quad \mathcal{B}_m H_{jk}^h = \lambda_m \mu_m g_{kh} - \left( \eta_m \right)_m H_{kr,h} \]

if and only if

\[ (2.12) C_{ijh} = 0 \]

since $\lambda_i$ and $\mu_m \neq 0$.

Thus, we conclude Theorem 2.2: In $G(\mathcal{B}) - R$ affinely connected space, if the directional derivative of covariant vectors field vanish, then Berwald curvature tensor $H_{hk}$ and its associative curvature tensor $H_{hk}$ are $GB - R$ if and only if the $(h)v$-torsion tensor $C_{ijh}$ vanishes.

Again, if $\delta \lambda_m = 0$ and $\delta \mu_m = 0$, then the equations (5) become

\[ \mathcal{B}_m H_{jk}^h = \lambda_m H_{jk}^h + (n - 1) \mu_m g_{kh} \]

and

\[ (2.13) \quad \mathcal{B}_m H_{jk}^h = \lambda_m H_{jk}^h + (n - 1) \mu_m g_{kh} \]

respectively. Hence, we conclude Theorem 2.3: In $G(\mathcal{B}) - R$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the $h$ Ricci tensor $H_{hk}$ and the tensor $(H_{hk} - H_{kh})$ are non-vanishing.

Also, if $\delta \lambda_m = 0$, then the equation (47) becomes

\[ \mathcal{B}_m H_{jk}^h = H_{jk}^h G_{m,n} + (\delta \lambda_m) H_{jk}^h + \lambda_m H_{jk}^h \]

respectively.

Thus, we conclude Theorem 2.4: In $G(\mathcal{B}) - R$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the $H - Ricci$ tensor $H_{hk}$ behaves as a recurrent.

We know that the Berwald curvature tensor satisfies the identity [11]

\[ B_m H_{jk}^h + B_k H_{jm}^h + B_k H_{jm}^h + H_{kl} G_{ijkl}^h + H_{kl}^h G_{ijkl}^h + H_{km} G_{ijkl}^h = 0. \]

Let us consider $G(\mathcal{B}) - R$ affinely connected space.

Suppose $\delta \lambda_m, \delta \mu_m = 0$ and by using (22) [5] in (2.16), we get

\[ (2.17) \quad \lambda_m H_{jk}^h + \lambda_k H_{mk}^h + \lambda_k H_{kh}^h + \mu_m (\delta^k y_h - \delta_k y_h) + \mu_k (\delta^k y_m - \delta_k y_m) + 2y^m \mu_m C_{jk}^m + 2y^m \mu_k C_{km}^m = 0. \]

Transvecting (2.17) by $y^p$, using (1.9), (1.11), (1.4), (1.1) and (1.6), we get

\[ (2.18) \quad \lambda_m H_{jk}^h + \lambda_k H_{mk}^h + \lambda_k H_{kh}^h + \mu_m (\delta^k y_h - \delta_k y_h) + \mu_k (\delta^k y_m - \delta_k y_m) + 2y^m \mu_m C_{jk}^m = 0. \]
curvature scalar $H$ are behave as recurrent tensors. Transvecting (2.19) by $g_{ip}$, using (2.3), (1.14), (1.1) and (1.5), we get
\[ B_m H_{kp,i} = H_{kp,i} + \mu_m (\gamma_x g_{kh} - \gamma_k g_p) . \]

Thus, we conclude

Theorem 2.8. In $G(BR) - R - \text{affine} - \text{connected} - \text{space}$, Berwald covariant derivative of first order for the associative torsion tensor $H_{kp,i}$ is given by (2.23). $I$ provided the directional derivative of covariant vectors field vanish $I$.

3. The Projection On Indicatrix In G(BR) - R - Affine Connected Space


Our aim is to discuss the projection on indicatrix for the $H - \text{Ricci}$ tensor $H_{kh}$ which behaves as recurrent in (BR) - $R - \text{affine} - \text{connected}$ - space.

The projection of any tensor $T^j_i$ on the indicatrix is given by
\[ T^j_i \rightarrow T^j_i - h^j_i h^i_j , \]
where
b) $h^j_i = |i|_j$.

The projection of the vector $y^i$, the unit vector $y^i$ and the metric tensor $g_{ij}$ on the indicatrix are given by
\[ a) p, y^i = 0, \]
\[ b) p, |i|_j = 0 \]
and
\[ c) p, g_{ij} = h_{ij} , \]
where
d) $h_{ij} = g_{ij} - |i|_j$.

Let us consider $G(BR) - R - \text{affine} - \text{connected}$ - space.

Since in $G(BR) - R - \text{affine} - \text{connected}$ - space, the $H - \text{Ricci}$ tensor $H_{kh}$ behaves as recurrent, i.e. satisfies the condition (2.15).

In view of (3.1a), the projection of the $H - \text{Ricci}$ tensor $H_{kh}$ is given by
\[ B_m H_{kh} = H_{kh} h^k_i h^i_k . \]

Taking the covariant derivative for the condition (3.3) with respect to $x^m$, in the sense of Berwald, we get
\[ B_m (p H_{kh}) = B_m (H_{kh} h^k_i h^i_k) . \]

Using the condition (2.15) and the fact that $h^k_i$ is covariant constant in (3.4), we get
\[ B_m (p H_{kh}) = \lambda_m (H_{kh} h^k_i h^i_k) . \]

Using (3.3) in (3.5), we get
\[ B_m (p H_{kh}) = \lambda_m (p H_{kh}) . \]

Thus, we conclude

Theorem 3.1. In $G(BR) - R - \text{affine} - \text{connected}$ - space, the projection of the $H - \text{Ricci}$ tensor $H_{kh}$ on indicatrix is recurrent.

Now, let us consider a Finsler space $F_B$ for which the projection of the $H - \text{Ricci}$ tensor $H_{kh}$ on indicatrix behaves as recurrent with respect to Berwald’s connection i.e. characterized by the condition (3.6).

Using (3.1a) in (3.6), we get
\[ B_m (H_{kh} h^k_i h^i_k) = \lambda_m (H_{kh} h^k_i h^i_k) . \]

Using (3.1b) in (3.7), we get
\[ B_m (H_{kh} h^k_i h^i_k) = \lambda_m (H_{kh} h^k_i h^i_k) . \]

Using (3.8) in (3.9), we get
\[ B_m (H_{kh} h^k_i h^i_k) = \lambda_m (H_{kh} h^k_i h^i_k) . \]

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