Hyperbolic KÄHler Manifold with Constant Holomorphic Sectional Curvature Tensor

B.B. Pandey
Swami Vivekanand College Of Management and Technology Gaulapar, Haldwani, (Nainital) Uttrakhand, India.

Abstract: Almost hyperbolic Hermite manifold have been studied by Dube [2]. Almost hyperbolic contact \( \{g, \eta, \xi\} \) structure manifold have been studied by Dube and Upadhyay [3]. In this present paper I have obtained some results regarding to the constancy of holomorphic sectional curvature for hyperbolic Kahler manifold.

1. Introduction
Let \( M^n \) be a \( C^\infty \) real differentiable manifold of dimension \( n \), \( (FM) \) be the ring of real valued differentiable function of \( M \) and \( (FM) \) be the module derivatives of \( (FM) \). Then \( (FM) \) is a Lie algebra over the real number and the elements of \( (FM) \) are called vector fields. Every Riemannian metric \( g \) associated with a general Riemannian manifold \( M \) defined an inner product in \( (FM) \), which we write as \( g(X,Y) \) for \( X,Y \in (FM) \), and let \( g \) be symmetric. Let \( M \) be equipped with a \( (1,1) \) tensor \( f \) which may be regarded as an \( (FM) \) linear map \( f:F(M) \rightarrow (FM) \) which satisfies

\[
F^2 X = X
\]

Where \( I \) is identity. Such a manifold is orientable and even dimensional. \( M \) is almost hyperbolic Hermitian provided it is locally product and has a Riemannian metric for which

\[
G(FX, FY) = g(X,Y)
\]

In hyperbolic Kahler manifold \( \{M,g,F\} \) is of constant holomorphic sectional curvature \( C(m) \) at every point \( m \). Then the Riemannian curvature tensor of \( M \), \( R(X,Y,Z,W) \) is of the form

\[
R(X,Y,Z,W) = C(m)/4[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,FW)g(Y,FZ) - g(X,FZ)g(Y,FW) - 2g(X,FY)g(Z,FW)]
\]

For any vector \( X,Y,Z,W \in TM \). In a hyperbolic Kahler manifold, we have

\[
R(X,Y,X,Y) = 1/32[3Q(X+FY)+3Q(X-FY)-Q(X+Y)-Q(X-Y)-4Q(X)-4Q(Y)].
\]

For any vector \( X,Y \in TM \) where

\[
Q(X) = R(X,FY,X,FY)
\]

Let \( (M,g,F) \) be a hyperbolic Kahler manifold, and let \( K(X,Y), R(X,Y,Z,W) \) and \( H(X) \) be the sectional curvature respectively for arbitrary vector \( X,Y,Z \) and \( W \) on \( M \) then the following identity

\[
R(X,Y,Z,W) - R(X,Y,FZ,FW) = 0
\]

Where \( f(X,Y) = g(FX) \)

**Theorem (1.1):** Let \( (M,g,F) \) be a hyperbolic Kahler manifold of constant holomorphic sectional curvature \( C(m) \) at every point \( m \) of \( M \), then the Riemannian curvature tensor of \( M \) is of the form

\[
R(X,Y,Z,W) = C(m)/4[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,FW)g(Y,FZ) - g(X,FZ)g(Y,FW) - 2g(X,FY)g(Z,FW)]
\]

**Proof:** Since \( Q(X) = H(X) \| X \|^4 \), we have

a) \( Q(X+FY) = H(X+FY) \| (X+FY) \|^4 \)

\[
= H(X+FY)(g(X,X)+g(Y,Y))^2 + 4g(X,X)+g(Y,Y)g(X,FY)
\]

(1.6) b) \( Q(X-FY) = H(X-FY) \| (X-FY) \|^4 \)
Corollary (2.3): Let (M,g,F) be an Quasi hyperbolic Kahler manifold of constant holomorphic sectional curvature C(m) at every point m of M . Then the Riemannian Curvature tensor of m is of the form :

\[ R(X,Y,Z,W) = C(m)/4[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)] - 2g((D_Xg)Y,(D_FZ)W) + 1/2[g((D_Xg)Y,(D_FZ)W) - 2g((D_Xg)Y,(D_FZ)W)] + 1/8[2g((D_Xg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F]

When (M,g,F)reduced to a Nearly hyperbolic Kahler manifold ,then we have

\[ (D_Xg)(Y) = -(D_Fg)X \]

In this case we have the following

Corollary (2.1): If a nearly hyperbolic Kahler manifold (M,g,F) is of constant holomorphic sectional curvature C(m) at m∈M,then R(X,Y,Z,W)is the form :

\[ R(X,Y,Z,W) = C(m)/4[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)] - 2g((D_Xg)Y,(D_FZ)W) + 1/2[g((D_Xg)Y,(D_FZ)W) - 2g((D_Xg)Y,(D_FZ)W)] + 1/8[2g((D_Xg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F]

If (M,g,F) is almost hyperbolic Kahler that is

\[ (D_Xg)(Y) + (D_Fg)(Z) + (D_Dg)F)(X) = 0 \]

then we have

Corollary (2.2): Let (M,g,F) be an almost Kahler manifold of constant holomorphic sectional curvature C(m) at every point m of M,then the Riemannian curvature tensor of m is of the form :

\[ R(X,Y,Z,W) = C(m)/4[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)] + g((D_Xg)Y,(D_FZ)W) + 1/8[2g((D_Xg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F]

-1/16[2g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F]

-1/16[2g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F] + g((D_Dg)F)(D_Fg)Z(D_Dg)F]
REFERENCE:

1. Bhatt, L. and Dube, K.K.: Hypersurface of hyperbolic Hermitian manifold-I, II (comm..)