The New Weibull Lomax Distribution

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Abstract: A three parameter blend of Weibull and Lomax distribution (NWLD) is introduced in this study. Reliability analysis and structural properties of the new distribution is studied which include survival, hazard, reverse hazard, cumulative hazard functions, moments about origin, mean deviations and quantiles. In addition, the expressions for Shannon entropy, order statistics are also developed. The model is being applied on real data set to assess its flexibility and usefulness under the approach of maximum likelihood estimation.

1. Introduction

The diversity found in the forms of lifetime distributions like Exponential, Weibull, Rayleigh, Modified Weibull, and Linear Exponential distribution is huge. Several distributions are developed for the modeling life time data. The Weibull distribution is one of the popular models that are used for modeling purposes. There exist broad aspects of the use of different forms of Weibull models. The Lomax distribution was proposed by Lomax [5] which is used widely for many applications in different fields such as medical, business failure life, actuarial sciences, wealth inequalities etc. Harris [3] has used Lomax distribution to model income and wealth. However, various methods are developed for the construction of probability models. In this study, a new probability model is developed as a generalization of Lomax distribution. The formulation of model is under the method as follows given by Al-Kadim and Boshi [1]. According to this criteria

2. The NWL Distribution

The New Weibull Lomax Distribution (NWLD) was developed by using the following criteria

$$G(x) = \int_0^{1/R_2(x)} f_1(x) dx$$ (1)

By substituting the values of $R_2(x)$ and $f_1(x)$, after simplification

$$G(x) = 1 - \exp(-\lambda \left(\frac{\beta}{x+\beta}\right)^{-\alpha \delta})$$

By differentiating equation (1), the pdf is

$$g(x) = \lambda \left(\frac{\alpha \delta}{\beta} \frac{(\beta}{x+\beta}\right)^{-\alpha \delta} \exp(-\lambda \left(\frac{\beta}{x+\beta}\right)^{-\alpha \delta})$$

; $x \geq -\beta, \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0$

Let $k = \lambda \delta$ and $\delta = \theta \alpha$, then cdf and pdf is of the form

$$G(x) = 1 - e^{-k (\frac{\beta}{x+\beta})^{-\delta}}$$ (2)

$$g(x) = k \left(\frac{\beta}{x+\beta}\right)^{-\delta-1} e^{-k (\frac{\beta}{x+\beta})^{-\delta}}$$ (3)

; $x \geq -\beta, \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0$

Lemma 2.1: The limits pdf of NWLD for $x \to +\infty$ is 0, for $x \to -\beta$ is 0

Proof:

$$\lim_{x \to +\infty} g(x) = 0, \lim_{x \to -\beta} g(x) = 0$$

Lemma 2.2: The limits of cdf of NWLD, for $x \to +\infty$ is 1, for $x \to -\beta$ is 0

Proof:

$$\lim_{x \to +\infty} G(x) = 1, \lim_{x \to -\beta} G(x) = 0$$

Lemma 2.3: The area under the curve is unity.
Proof:
\[
\int_{-\beta}^{\infty} y(x) \, dx = \int_{-\beta}^{\infty} k\left(\frac{x}{\alpha+\beta}\right) e^{-k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1}} \, dx
\]

Let \( y = k\left(\frac{x}{\alpha+\beta}\right)^{\beta} \) then, \( dy = k\left(\frac{\beta}{\alpha+\beta}\right) \left(\frac{x}{\alpha+\beta}\right)^{\beta-1} \, dx \)
when \( x = -\beta \) then \( y = 0 \)
when \( x = +\infty \) then \( y = +\infty \)

\[
x = \beta \left(\frac{y}{k}\right)^{\frac{1}{\beta}} - 1
\]

\[
\int_{-\beta}^{\infty} y(x) \, dx = \int_{0}^{\infty} e^{-z^2} \, dz = (-1)(e^{-z} - e^{\beta})
\]

\[
\int_{-\beta}^{\infty} g(x) \, dx = 1
\]

The plots of pdf and cdf of NWLD in Figure 1 and Figure 2 for selected parametric values are demonstrated respectively.

![Figure 1: Probability Density Function of NWLD](image1)

![Figure 2: Cumulative Distribution Function of NWLD](image2)

3. Reliability Analysis

In this section various measures of reliability of NWLD are derived.

The survival function NWLD is
\[
S(x) = 1 - G(x) = e^{-k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1}}
\]

(4)

The Hazard function NWLD is
\[
h(x) = \frac{k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1} e^{-k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1}}}{1 - e^{-k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1}}}
\]

(6)

The Cumulative Hazard function NWLD is
\[
H_x(x) = \ln[5(x) + 1]
\]

(7)

4. Statistical Properties

In this section we study properties of NWLD

4.1. Moments

The \(r^{th}\) moment of NWLD is
\[
E(X^r) = \sum_{i=0}^{\infty} \binom{r}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-1} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(8)

For \( r = 1, 2, 3, 4 \)

\[
\mu_1 = E(X) = \sum_{i=0}^{\infty} \binom{1}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-1} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(9)

\[
\mu_2 = E(X^2) = \sum_{i=0}^{\infty} \binom{2}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-2} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(10)

\[
\mu_3 = E(X^3) = \sum_{i=0}^{\infty} \binom{3}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-3} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(11)

\[
\mu_4 = E(X^4) = \sum_{i=0}^{\infty} \binom{4}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-4} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(12)

4.2. Moment Generating Function

The moment generating function (m.g.f.) of NWLD is
\[
M_x(t) = E(e^{tx}) = \int_{-\beta}^{\infty} e^{tx} \cdot g(x) \, dx
\]

we have,
\[
M_x(t) = \int_{-\beta}^{\infty} e^{tx} \cdot g(x) \, dx
\]

Now, using the expansion of \(e^{tx}\), we have
\[
e^{tx} = \sum_{v=0}^{\infty} \frac{(t)^v x^v}{v!}
\]

Therefore we have the expression as,
\[
M_x(t) = \sum_{v=0}^{\infty} \frac{(t)^v x^v}{v!} \int_{-\beta}^{\infty} x^v \cdot g(x) \, dx
\]

\[
= \sum_{v=0}^{\infty} \frac{(t)^v}{v!} \cdot E(X^v)
\]

(13)

\[
M_x(t) = \sum_{v=0}^{\infty} \frac{(t)^v}{v!} \cdot \sum_{i=0}^{\infty} \binom{v}{i} \left(\frac{\beta}{\gamma}\right)^i \left(-\beta\right)^{i-1} \Gamma\left(\frac{i}{\beta} + 1\right)
\]

(14)

4.3. Mean Deviations

The mean deviation by mean and median of NWLD respectively is given by
\[
D(\mu) = 2(\mu + \beta) \left[1 - e^{-k\left(\frac{x}{\alpha+\beta}\right)^{\beta+1}}\right] - 2\beta x
\]

(15)
\[ D(m) = \mu - 2\beta k^{-1\gamma} \left( \frac{1}{\beta} + 1, k\left(\frac{m+\delta}{\beta}\right)^{\gamma} + 2\beta \right) \left(1 - e^{-k\left(\frac{m+\delta}{\beta}\right)^{\gamma}}\right) \]  
(16)

### 4.4. Median

The Median of NWLD is

\[ \text{Median}(x) = x_{0.5} = \beta \left( \frac{1}{\frac{1}{\beta} \ln(2)} \right)^{\frac{1}{\gamma}} - 1 \]  
(17)

### 4.5. Quantile Function

The quantile function for NWLD is

\[ Q(p) = \beta \left( \frac{1}{\frac{1}{\beta} \ln \left( \frac{1}{1-p} \right) \gamma} \right)^{\frac{1}{\gamma}} - 1 \]  
(18)

### 4.6. Random Number Generator

The random number generator for NWLD is

\[ x = \beta \left( \frac{1}{\frac{1}{\beta} \ln \left( \frac{1}{1-u} \right) \gamma} \right)^{\frac{1}{\gamma}} - 1 \]  
Where \( u \sim U(0,1) \)  
(19)

### 4.7. Shannon Entropy

The Shannon Entropy of NWLD is

\[ E[-\log g(x)] = -\log \left( \frac{k\alpha}{\beta} \right) + \left(1 - \frac{1}{\alpha}\right) y + 1 \]  
(20)

Where \( y \) is Euler Gamma Constant.

### 5. Order Statistics

The distribution of \( s^{th} \) order statistic of NWLD is

\[ g_{s}(x) = \frac{1}{(s-1)!} g(x) g(\alpha)^{s-1} \left(1 - g(x)\right)^{n-s} \quad ; s = 1,2,3,\ldots \]  
(21)

Then the distribution of smallest order statistic of NWLD is

\[ g_{1}(x) = nk\left(\frac{1}{\frac{1}{\beta} \ln(\alpha)}\right)^{s} e^{-k\left(\frac{1}{\beta} \ln(\alpha)\right)^{s}} \left[1 - e^{-k\left(\frac{1}{\beta} \ln(\alpha)\right)^{s}}\right]^{n-1} \]  
(22)

Then the distribution of largest order statistic of NWLD is

\[ g_{n}(x) = nk\left(\frac{1}{\frac{1}{\beta} \ln(\alpha)}\right)^{s} e^{-k\left(\frac{1}{\beta} \ln(\alpha)\right)^{s}} \left[1 - e^{-k\left(\frac{1}{\beta} \ln(\alpha)\right)^{s}}\right]^{n-1} \]  
(23)

### 6. Maximum Likelihood Approach

The maximum likelihood approach is used for estimating parameters. The Log-Likelihood function is given by

\[ \ln L = n \ln(k) + n \ln(\beta) + (\delta - n + 1) \ln(\delta) - (\delta + 1) \sum_{i=1}^{n} \ln(x_i + \beta) - k\beta^\delta \sum_{i=1}^{n} (x_i + \beta)^{-\delta} \]  
(24)

Now differentiate log-likelihood with respect to \( k, \delta \) and \( \beta \) respectively we obtain

\[ \frac{\partial \ln L}{\partial k} = \frac{n}{k} - \beta^\delta \left( \sum_{i=1}^{n} (x_i + \beta)^{-\delta} \right) \]  
(25)

\[ \frac{\partial \ln L}{\partial \delta} = \frac{n}{k} - \ln(\delta) - \sum_{i=1}^{n} \ln(x_i + \beta) - k\beta^\delta \sum_{i=1}^{n} (x_i + \beta)^{-\delta} \]  
(26)

\[ \frac{\partial \ln L}{\partial \beta} = \frac{\delta - n + 1}{\beta} - (\delta + 1) \sum_{i=1}^{n} (x_i + \beta)^{-\delta} - k\beta^\delta \sum_{i=1}^{n} (x_i + \beta)^{-\delta} \]  
(27)

The estimates of maximum likelihood are obtained via solving above equations by non-linear equation system. Iterative method developed by Newton Raphson is used to solve the equations. According to the large sample approximation, the estimators of the maximum likelihood approach can be treated as multivariate normal approximately. Therefore, when \( n \to \infty \), the asymptotic distribution of \( \alpha, \beta \) and \( \lambda \) is as follows

\[ \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \overset{MVN}{\rightarrow} \begin{pmatrix} \frac{\partial g(x)}{\partial \alpha} \\ \frac{\partial g(x)}{\partial \beta} \\ \frac{\partial g(x)}{\partial \lambda} \end{pmatrix} \]  
(28)

Where \( V_{ij} = V_{ij} \left(\frac{\partial g(x)}{\partial \alpha}, \frac{\partial g(x)}{\partial \beta}, \frac{\partial g(x)}{\partial \lambda} \right) \) is the approximate Variance-Covariance Matrix which can be approximately determined by using the Information Matrix

\[ V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = I_{11} I_{12} I_{13} \\ I_{21} I_{22} I_{23} \\ I_{31} I_{32} I_{33} \]  
(29)

Where,

\[ I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \]  
is the Information Matrix.

For NWLD, all the second order derivatives exist. Hence we can obtain

\[ I_{11} = \frac{\partial^2 \ln L}{\partial k^2}, \quad I_{12} = I_{21} = \frac{\partial^2 \ln L}{\partial k \partial \delta}, \quad I_{13} = I_{31} = \frac{\partial^2 \ln L}{\partial k \partial \lambda} \]  
\[ I_{22} = \frac{\partial^2 \ln L}{\partial \delta^2}, \quad I_{23} = I_{32} = \frac{\partial^2 \ln L}{\partial \delta \partial \lambda} \]  
\[ I_{33} = \frac{\partial^2 \ln L}{\partial \lambda^2} \]  
(30)

By solving the Fisher Matrix, we find variance-covariance matrix. Hence, an approximate 100(1 - \( \delta \))\% confidence interval for \( \alpha, \beta \) and \( \lambda \) is given by
Where \( z_{\frac{\alpha}{2}} \) is the upper \( \frac{\alpha}{2} \) percentile of standard normal distribution.

7. Application

In this part, we apply NWLD on a lifetime data set. The data set (gauge length of 10mm) from Kundu and Raqab[4].The data set holds sixty-three observations as:


We compare NWLD with the following four distributions

1) Lomax Distribution(LD)
   \[ f(x) = \frac{\beta}{(1 + \lambda x)^{\beta + 1}} : x > 0, \ \lambda, \beta > 0 \]

2) Rayleigh Distribution (RD)
   \[ f(x) = \frac{x}{\beta} \exp \left( - \frac{x^2}{2\beta} \right) : x > 0, \ \beta > 0 \]

3) Exponential Distribution (ED)
   \[ f(x) = e^{-\beta x} : x > 0, \ \beta > 0 \]

4) Exponential Lomax Distribution(ELD)
   \[ f(x) = \frac{e^{-\beta x}}{\beta} \left( \frac{\beta}{\beta + \lambda x} \right)^{-\lambda - 1} \exp \left( - \frac{\beta x}{\beta + \lambda x} \right) : x > 0, \lambda, \beta > 0 \]

The ML estimates of five distributions are computed with their respective standard errors. The six criteria’s used for comparison are

a) \[ AIC = -2LL + 2P \]

b) \[ BIC = -2LL + P \log (n) \]

c) \[ CAIC = -2LL + 2Pn/(n - P - 1) \]

d) Kolmogorov – Smirnov Test(KS)

Where:

\[ P = \text{Number of Parameters}, \]
\[ n = \text{Number of Observations} \]

Table 1 shows that the smallest value of goodness of fit measures is given by NWLD for all criteria’s among all models under study. Figure 7 shows the empirical cumulative distribution graph for estimated values of ED, RD, LD, NWLD and ELD.

8. Conclusion

In this study we have derived a new form of Weibull-Lomax distribution namely New Weibull Lomax Distribution (NWLD). The distribution is found to be more flexible from the parent and related probability models. We derived mathematical properties of the distribution and a view is given on the order statistics and record values of the proposed model. The new
distribution applied on the data set is much better than other models.

9. References


