Some Results on Group Unitary Order Labeling Of Graphs

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Abstract: The vertices of a graph can be labeled in many ways suiting to the situations arising out of applications. The notion of unitary order labeling (UOL) was introduced by B.D. Acharya\textsuperscript{1}. It is labeling the vertices of a graph \(G = (V,E)\) with elements of a commutative group \((\Gamma,\ast)\) in an injective manner such that 
\[
(o(f(u) \ast f(v)), |V(G)|) = 1 \quad \text{for all } uv \in E(G).
\]
He also defined Group unitary order potential graph as a graph which admits UOL. In this paper based on these definitions, we have proved some results on GUOP graph. Edge analog of this concept has been introduced.

1. Introduction

Algebraic graph theory is concerned with the study of abstract algebraic structures using graph theoretical tools and techniques. As we are interested in this, the concept of group unitary order labeling introduced by Dr. B.D. Acharya motivated us to work on it.

Definition 1.1

Let \((\Gamma,\ast)\) be a commutative group and \(G = (V,E)\) be any graph\textsuperscript{2,4}. Then a \(\Gamma\)-unitary order labeling is an injective function \(f : V(G) \rightarrow \Gamma\) such that the following condition is satisfied
\[
uv \in E(G) \Rightarrow (o(f(u) \ast f(v)), |V(G)|) = 1.
\]

Definition 1.2

A graph \(G\) is said to be \(\Gamma\)-unitary order potential (GUOP) if there exists a \(\Gamma\)-unitary order labeling for \(G\).

Example 1.3

Consider \(Z_5 = \{0, 1, 2, 3, 4\}\). \(\Gamma = (Z_5, \oplus)\) is a commutative group. Consider \(G = C_5\). Define \(f : V(G) \rightarrow \Gamma\) such that the following condition is satisfied
\[
uv \in E(G) \Rightarrow (o(f(u) \oplus f(v)), |V(G)|) = 1.
\]

Theorem 1.4

Any graph on \(n\) vertices admits unitary order labeling with respect to the group \((Z_{n+1}, \oplus)\).

Proof

Let \(G\) be an arbitrary graph with \(n\) vertices. Let \(\Gamma = (Z_{n+1}, \oplus)\). Let \(f\) be an injective function from \(V(G)\) to \(Z_{n+1}\). For any edge \(uv \in E(G)\), consider \(f(u) \oplus f(v)\). The order \(o(f(u) \oplus f(v))\) divides \(n+1=|Z_{n+1}|\).
\[
(n,n+1)=1 \Rightarrow (o(f(u) \oplus f(v)), n) = 1, \forall uv \in E(G).
\]

Any graph is a GUOP graph.

Theorem 1.5

Let \(\Gamma = (P(S), \Delta)\) where \(|S| = n\).

i) If \(\gamma(G)\) is odd and \(\gamma(G) \leq 2^n\), then \(G\) is \(\Gamma\)-UOP.

ii) If \(\gamma(G)\) is even and \(G\) is \(\Gamma\)-UOP then \(G\) is 1-regular and \(\gamma < 2^n - 1\).

Proof

i) Order of every element in \(P(S)\) other than the empty set is 2. So any injective map \(f : V(G) \rightarrow \Gamma\) results in \(\Gamma\)-UOL.

ii)\(\left(o(f(u) \Delta f(v)), \gamma(G)\right) = 1\)
\[
\Rightarrow o(f(u) \Delta f(v)) = 1 \quad \text{(Because } \gamma(G) \text{ is even)}
\]
\[
\Rightarrow f(u) \Delta f(v) = \varphi
\]

If \(uv \in E(G)\) then in any \(\Gamma\) - UOL, \(f(u)\) and \(f(v)\) are inverses of each other. Hence \(\deg(u) = 1 \quad \forall u \in G\). So \(G\) is 1-regular. Since \(\varphi^{-1} = \varphi\), \(\varphi\) cannot be assigned to any vertex in \(G\). Hence \(\gamma < 2^n - 1\).
Theorem 1.6
Let $\Gamma = \left( Z_p^k, \oplus \right)$ where $p$ is prime and $p \neq 2$. A graph $G$ with $(\gamma(G), p) \neq 1$ is $\Gamma$ - UOP iff it is a subgraph of $\left( \frac{n}{2} \right)_2 K_2$.

Proof
Let $G = (V, E)$ be a graph with $\gamma$ vertices. Since $\Gamma$ is a finite group, $o(a)$ is a divisor of $o(\Gamma)$ for all $a \in \Gamma$. Since $o(\Gamma) = p^k$ and $p$ is a prime,

$$\frac{p}{o(a)} \in \Gamma \implies o(a) = p^l, \, 0 \leq l \leq k.$$ 

The identity element has order 1 and the other elements have order $p^l$, $l \geq 1$.

$(\gamma, p) \neq 1 \implies \gamma$ is a multiple of $p$. $G$ is $\Gamma$ - UOP if and only if

$$\left( o(f(u) \oplus f(v)), |V(G)| \right) = 1 \forall uv \in E(G)$$

This is possible only when $l = 0$ (i.e., when $f(u) \oplus f(v) = 0$). That is when if $f(u) = [f(v)]^{-1}$ and $f(v)$ are inverses of each other. Hence $\deg(u) = 1$ and $\deg(v) = 1$ for all $G$ is $\Gamma$ - UOP. So in any $\Gamma$ - UOP graph, $E(G)$ should be the disjoint union of at most $\left( \left( \frac{n}{2} \right)^{-1} - 1 \right)_2 K_2$. Hence $G$ is $\Gamma$ - UOP if it is a subgraph of $\left( \frac{n}{2} \right)_2 K_2$.

Note.
When $p = 2$, $G$ is $\Gamma$ - UOP iff $G \subseteq \left( \left( \frac{n}{2} \right) - 1 \right)_2 K_2$, since $0^{-1} = 0$ and $\frac{n}{2}$ is in $Z_n$.

Theorem 1.7
Let $\Gamma = \left( Z_n, \oplus \right)$. No graph with $\gamma(G) = n$ is $\Gamma$ - UOP.

Proof
Since $\delta(G) \geq 1$, $G$ has no isolated vertices. If possible let $G$ be $\Gamma$ - UOP. Then

$$\left( o(f(u) \oplus f(v)), n \right) = 1, \forall uv \in E(G)$$

$$o(a)/n, \forall a \in \Gamma$$

$$\Rightarrow o(f(u) \oplus f(v)) = 1, \forall uv \in E(G)$$

$$\Rightarrow f(u) \oplus f(v) = 0, \forall uv \in E(G)$$

$$f(u) = [f(v)]^{-1}, \forall uv \in E(G)$$

Since UO is an injective function, 0 cannot be given to any vertex in $G$.

$\gamma(G) < n$ which is a contradiction. Hence no graph with $\gamma(G) = n$ is $\Gamma$ - UOP.

Theorem 1.8
Let $\Gamma = \left( Z_{2.3^k}, \oplus \right), n = 2.3^k, \, k \geq 1$. Let $G$ be any $\Gamma$ - UOP graph with $\gamma$ vertices $(\gamma < n)$.

1. If $(\gamma, 2) = 1$ and $(\gamma, 3) = 1$, then $G$ can be any graph.

2. If $(\gamma, 2) = 1$ and $(\gamma, 3) \neq 1$, $G$ is a subgraph of $\left( \frac{n}{2} - 1 \right)_2$ copies of $K_2$.

3. If $(\gamma, 2) \neq 1$ and $(\gamma, 3) = 1$, then $G$ is a subgraph of union of two complete graphs on $3^k$ vertices.

If $(\gamma, 2) \neq 1$ and $(\gamma, 3) \neq 1$, $\gamma < n$ then $G$ is a subgraph of $\left( \frac{n}{2} - 1 \right)_2$ copies of $K_2$.

Proof
Let $a \in \Gamma$.

$$o(a) = \begin{cases} 
1 & \text{if } a = 0 \\
2 & \text{if } (a, n) = 1 \\
\frac{n}{2} & \text{if } 2^k/a \, \text{and } 3^k,a \\
2.3^{k-1} & \text{if } 2^k/a \, \text{and } 3^{k-1}/a \\
3^{k-1} & \text{if } 2^k/a \, \text{and } 3^{k-1}/a 
\end{cases}$$

Case 1
$(\gamma, 2) = 1, \, (\gamma, 3) = 1$.

As $o(\Gamma) = 2.3^k$, $(o(a), \gamma) = 1$ for any $a \in \Gamma$.

Case 2
$(\gamma, 2) = 1, \, (\gamma, 3) \neq 1$.

For UO, admissible orders for $f(u) \oplus f(v)$ are 1 and 2 and admissible sums are 0, $\frac{n}{2}$. Among the
2×3^k vertices, we can join vertices having labels so that their sums are 0 (or) \( \frac{n}{2} \). In the resulting graph \( H_1 \), every vertex except 0, \( \frac{n}{2} \) will have degree 2 and 0, \( \frac{n}{2} \) will have degree 1. \( H_1 \) will be the disjoint union of \( k \) and \( \frac{3^k-1}{2} \) copies of \( C_4 \). Any sub graph of \( H_1 \) will be \( \Gamma-\text{UOP} \) and vice versa.

Case 3

\((\gamma, 2) \neq 1 \) and \((\gamma, 3) = 1\)

For UOL, admissible orders for \( f(u) \oplus f(v) \) are 1, 3, 3^2, ... 3^k and admissible sums are 0, 2, 4, 6, 8, 10, 14, ... 2(3^k-2). Among the \( 2\times3^k \) vertices, we can join vertices having labels so that their sums are 0, 2, 4, 6, ... 2(3^k-2). In the resulting graph \( H_2 \), every vertex is of degree 3^k. All odd vertices will form one complete graph and all even vertices will form another complete graph. \( H_2 \) will be the disjoint union of two complete graphs on 3^k vertices. So in this case G will be a sub graph of \( H_2 \) (i.e.,) the disjoint union of two copies of \( K_{3^k} \).

Case 4

\((\gamma, 2) \neq 1 \), \((\gamma, 3) \neq 1, \gamma \neq 1\)

For UOL, admissible order for \( f(u) \oplus f(v) \) is 1 and the admissible sum is 0. Among the \( 2\times3^k \) vertices, we can join vertices having labels so that their sum is 0 (i.e.) we can join the vertices having labels which are inverses of each other. 0 and \( \frac{n}{2} \) have degree 0, all other vertices have degree 1. The resulting graph \( H_3 \) will be the disjoint union of \( 3^k-1 \) copies of \( K_3 \). G will be a sub graph of \( H_2 \) (i.e.,) the disjoint union of \( 3^k-1 \) copies of \( K_3 \).

Illustration

Let \( \Gamma = (\mathbb{Z}_{18}, \oplus) \), 18=2.3^2.

<table>
<thead>
<tr>
<th>Elements of ( \mathbb{Z}_{18} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>18</td>
<td>18</td>
<td>9</td>
<td>9</td>
<td>18</td>
<td>3</td>
<td>18</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Elements of ( \mathbb{Z}_{18} )</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>2</td>
<td>9</td>
<td>18</td>
<td>3</td>
<td>18</td>
<td>9</td>
<td>6</td>
<td>9</td>
<td>18</td>
</tr>
</tbody>
</table>

The following table gives all possible UOL for various values of \( \gamma \) (\( \gamma < 18 \)).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Admissible Orders</th>
<th>Admissible Sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 5, 7, 11, 13, 17</td>
<td>Any order</td>
<td>Any sum</td>
</tr>
<tr>
<td>9, 15</td>
<td>1, 2</td>
<td>0, 9</td>
</tr>
</tbody>
</table>

H_1: (admissible sums 0, 9)

H_1 is the disjoint union of \( K_2 \) and \( 4C_4 \).

H_2: (admissible sum 0)

H_2 is the disjoint union of 8 copies of \( K_3 \).

H_3: (admissible sums 0, 2, 4, 6, 8, 10, 12, 14, 16)

H_3 is the disjoint union of two copies of \( K_3 \).
Section 2.

Situations may demand labeling with non-abelian group also[3]. GUOL can be extended to any group as follows:

Definition 2.1
Let \((\Gamma, \ast)\) be any group and \(G = (V, E)\) be any graph. A \(\Gamma\)-unitary like order labeling is an injective function \(f : V(G) \rightarrow \Gamma\) satisfying the following conditions:

i) \(uv \in E(G) \Rightarrow (o(f(u) \ast f(v)), |V(G)|) = 1\)
ii) \(uv \in E(G) \Rightarrow (0(f(v) \ast f(u)), |V(G)|) = 1\)

Definition 2.2
Let \((\Gamma, \ast)\) be any group and \(G = (V, E)\) be any graph. A weak \(\Gamma\)-unitary order labeling is an injective function \(f : V(G) \rightarrow \Gamma\) such that

i) \(uv \in E(G) \Rightarrow (o(f(u) \ast f(v)), |V(G)|) = 1\)

or

ii) \(uv \in E(G) \Rightarrow (0(f(v) \ast f(u)), |V(G)|) = 1\)

Note
1. Every group unitary order labeling is a weak \(\Gamma\)-unitary order labeling
2. Every \(\Gamma\)-unitary like order labeling is a weak \(\Gamma\)-unitary order labeling but the converse need not be true.

Definition 2.3
A graph \(G\) is said to be \(\Gamma\)-unitary like order potential (GULOP) if there exists a \(\Gamma\)-unitary like order labeling for \(G\).

Dihedral group
A dihedral group is the group of symmetries of a regular polygon including both rotations and reflections. Dihedral groups play an important role in group theory, Geometry and Chemistry. \(D_n\) refers to group of the symmetries of a regular polygon with \(n\) sides. A regular polygon with \(n\) sides has \(2n\) different symmetries (\(n\) rotational symmetries and \(n\) reflection symmetries). The associated rotations and reflections make up the dihedral group \(D_n\). If \(n\) is odd, each axis of symmetry connects the mid point of one side to the opposite vertex. If \(n\) is even there are \(\frac{n}{2}\) axes of symmetry connecting the mid points of opposite sides and \(\frac{n}{2}\) axes of symmetry connecting opposite vertices. In either case there are \(n\) axes of symmetry altogether and \(2n\) elements in the symmetry group. In general, the group \(D_n\) has elements \(R_0, R_1, \ldots R_{n-1}\) and \(S_0, S_1, \ldots S_{n-1}\) with composition given by the following formulae:

\[
R_iR_j = R_{i+j}, \quad R_iS_j = S_{i+j}, \quad S_iR_j = S_{i-j}, \quad S_iS_j = R_{i-j}
\]

In all cases, addition and subtraction of subscripts should be performed using modular arithmetic with modulo \(n\).

Theorem 2.3
Let \(D_n\) be the \(n\)th dihedral group where \(n\) is odd. Let \(\Gamma = (\Gamma, \circ)\) where \(\circ\) denotes composition of permutations. Let \(G\) be any \(\Gamma\)-ULOP graph with \(\gamma < 2n\) vertices.

i) If \((\gamma, 2n) = 1\), then \(G\) can be any graph

ii) If \((\gamma, 2) \neq 1\), then \(G\) is a sub-graph of disjoint union of two complete graphs on \(n\) vertices.

Proof
Let \(a \in \Gamma\)

\[o(a) = \begin{cases} 1 \text{ if } a = R_0, \text{the identity element} \\ n \text{ if } a \text{ is a rotation} \\ 2 \text{ if } a \text{ is a reflection} \end{cases}\]

Case 1.
If \((\gamma, 2n) = 1\). Any graph on \(\gamma\) vertices admit UOL

Case 2. \((\gamma, 2) \neq 1\)

For UOL, admissible orders for \(f(u) \oplus f(v)\) are 1 and \(n\), admissible products are \(R_0, R_1, R_2 \ldots R_n\). As product of two rotations is a rotation and product of two reflections is a rotation, we can join vertices having labels so that their products are \(R_0, R_1, R_2 \ldots R_n\). All rotations will form one complete graph and all reflections will form another complete graph. The resulting graph \(H_1\) is the disjoint union of two complete graphs on \(n\) vertices.

Illustration 2.4
\[
D_5 = \{R_0, R_1, R_2, R_3, R_4, S_0, S_1, S_2, S_3, S_4\}
\]

The following table gives all possible UOL for various values of \(\gamma (\leq 10)\):

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>Admissible Orders</th>
<th>Admissible products</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 3, 7, 9</td>
<td>Any order</td>
<td>Any product</td>
</tr>
<tr>
<td>2, 4, 6, 8</td>
<td>1, 5</td>
<td>(R_0, R_1, R_2, R_3, R_4)</td>
</tr>
<tr>
<td>5</td>
<td>1, 2</td>
<td>(R_0, S_0, S_1, S_2, S_3, S_4)</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>(R_0)</td>
</tr>
</tbody>
</table>
H1: (Admissible product (R₀, R₁, R₂, R₃, R₄))

H₁ is the disjoint union of two complete graphs on n vertices.

H2: (Admissible product is R₀)

H₂ is disjoint union of two copies of K₂.

Theorem 2.4

Let (D₂ⁿ, o), n ≥ 1 where D₂ⁿ is the dihedral group with 2ⁿ⁺¹ elements. Let G be any Γ-ULOP graph with γ(2ⁿ⁺¹) vertices.

(i) If (γ, 2ⁿ⁺¹) = 1 then G can be any graph

(ii) If (γ, 2) ≠ 1, then G is a sub graph of \(2ⁿ⁻¹ - 1\)K₂

Proof

Let a ∈ Γ. \(o(a) =

1 if a = R₀, the identity element

an even number if ‘a’ is a rotation

2 if ‘a’ is a reflection

Case 1. (γ, 2ⁿ⁺¹) = 1

In this case for Γ-ULOP, any order for \(f(u) o f(v)\) is admissible. Hence, any graph admits Γ-ULOP.

Case 2. (γ, 2) ≠ 1

For Γ-ULOP, admissible order for \(f(u) o f(v)\) is 1 and the admissible product is R₀.

For any \(uv ∈ E(G)\),

\(f(u) o f(v) = R₀ \Rightarrow f(u) - f(v)⁻¹\)

The elements \(R₀, S₁, S₂, S₃, S₄, ..., S_{2n-1}\) have self inverses. Among the 2ⁿ⁺¹ vertices, we can join the vertices having labels so that their products \(R₁R₃, R₂R₄, R₃R₂, ..., R_{2n-3}R_{2n-1}\) is R₀. The resulting graph H is the disjoint union of \(2ⁿ⁻²\) copies of K₂. Hence, G is a sub graph of \(2ⁿ⁻¹ - 1\)K₂.

Theorem 2.5

Consider the Quaternion group \(\Gamma = Q\). A graph \(G\) with γ(8) vertices is Γ-U L O P if it is a sub-graph of three copies of K₂.

Proof

\(Q = \{±1, ±i, ±j, ±k\}\)

The following table gives all possible ULOL for various values of γ(≤ 8)

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>-1</th>
<th>i</th>
<th>-i</th>
<th>j</th>
<th>-j</th>
<th>k</th>
<th>-k</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Admissible Orders</td>
<td>Any order</td>
<td>Any product</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1, 3, 5, 7</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2, 4, 6, 8</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

H is disjoint union of three copies of K₂. Hence G is Γ-U L O P if it is a subgraph of three copies of K₂.

Section 3

Definition 3.1

Let (Γ, o) be a commutative group and G = (V, E) be any graph. Then, a Γ-unitary order edge labeling is an injective function \(f: E(G) → Γ\) such that

\(\prod_{e ∈ E(G)} f(e) \prod_{u ∈ V(G)} = 1, \forall (G, f) \in Γ\).

Definition 3.2

An arbitrary graph G is a group unitary order edge potential graph (GUOEP) if there exist a
commutative group $\Gamma$ and a $\Gamma$ unitary edge order labeling $f: E(G) \rightarrow \Gamma$.

Definition 3.3
Let G be any graph and $(\Gamma, \cdot)$ be a commutative group. A $\Gamma$-unitary order edge labeling is a strict $\Gamma$-unitary order edge labeling if
\[
\left\{ \begin{array}{l}
O\left( \prod_{e \text{adj. with } u} f(e) \right), E(G) \right\} = 1
\end{array} \right.
\]
\forall \text{ adjacent edges } e_i, e_j \in E(G)

Example 3.4
Consider $Z_6 = \{0,1,2,3,4,5\}$ which is an abelian group. Consider the graph $C_5$. Define $f: E(G) \rightarrow \Gamma$ such that $f(e_1) = 0$, $f(e_2) = 1$, $f(e_3) = 2$, $f(e_4) = 3$, $f(e_5) = 4$.

Theorem 3.5
Let $\Gamma = (P(S), \Delta)$ and $n = |S|$

(i) Any graph with odd number of edges $|E(G)| < 2^n$ admits $\Gamma$-UOEL.

(ii) If $|E(G)|$ is even and $\gamma(G) > 2$ then G never admits $\Gamma$-UOEL.

Proof
Let $|E(G)|$ be odd. Order of every element in P(S) other than the empty set is 2. Let G be any graph Choose S such that $2^{S+1} < |E(G)| < 2^S$. Let $K = |E(G)|$. Assign the elements of P(S) to the edges of $K^n$ in an injective manner. Clearly $O\left( \prod_{e \text{adj. with } u} f(e) \right), E(G) \right\} = 1$ for all adjacent edges $e_i, e_j \in E(G)$.

Hence $K_n$ admits strict $\Gamma$-UOEL if $n \equiv 2 \text{ or } 3 \pmod{4}$.

Conclusion
Usually we will start with some families of graphs and label them using certain numbers, sets or groups. But in GUOL, since each graph happens to be GUOP, we start with particular groups and search for graphs which are GUOP with respect to these groups. We can extend this work in finding more families of graphs which admit UOEL with respect to a given group according to the situations arising out of applications.

References
[1] Acharya, B.D., “some new directions of research in Algebraic Graph Theory”, ADMA- 2012, Pre-conference National Workshop on Algebraic Graph Theory.