Decomposability of Curvature Tensors in Non-Symmetric Recurrent Finsler Space

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Abstract. In this paper, we introduced a Finsler space $F_n^+$ for which the curvature tensors $R_{jkh}^+$ and $K_{jkh}^+$ are satisfied the generalized recurrence property with respect to non-symmetric connection parameter $(\Gamma_{jkh}^i \neq \Gamma_{kjh}^i)$ which given by the conditions

\[ R_{jkh1}^+ = \lambda_i R_{jkh}^+ + \mu_i (\delta^i_h g_{jk} - \delta^i_k g_{jh}) \]

and

\[ K_{jkh1}^+ = \lambda_i K_{jkh}^+ + \mu_i (\delta^i_h g_{jk} - \delta^i_k g_{jh}) \]

respectively, where $\lambda_i$ is the $v$-covariant differential operator, and $\mu_i$ are non-zero covariant vectors.

The purpose of this paper is to develop the above spaces by study the possibilities form of decompositions for the curvature tensors $R_{jkh}^+$ and $K_{jkh}^+$. We established the decompositions for the curvature tensors $R_{jkh}^+$ and $K_{jkh}^+$ in a Finsler space $F_n$ equipped with non-symmetric connection parameter.

Keywords. Generalized $R^+$ - recurrent space, Generalized $K^+$ - recurrent space, Decomposition of the curvature tensor $R_{jkh}^+$ and Decomposition of the curvature tensor $K_{jkh}^+$

1. Introduction

C. K. Mishra and G. Lodli [4] discussed $C^h$ - recurrent and $C^v$ - recurrent Finsler space of second order and obtained different theorems regarding these spaces, also discussed the decomposability of the curvature tensor in recurrent conformal Finsler spaces. The decomposability of the curvature tensor $R_{jkh}^+$ in recurrent Finsler space equipped with non-symmetric connection parameter studied by P. Mishra, K. Srivistava and S. B. Mishra [5].

Let us consider an n-dimensional Finsler space $F_n$ equipped with the metric function $F$ satisfying the requisite condition [7]. Let consider the components of the corresponding metric tensor $g_{ij}$, Cartan’s connection parameter $\Gamma_{jkh}^i$. These are symmetric in their lower indices and positively homogeneous of degree zero in directional argument. The two sets of quantities $g_{ij}$ and its associative tensor $g^{ij}$, which are related by

\[ g_{ij} g^{jk} = \delta^j_i, \text{ if } i = k, \]

\[ g_{ij} g^{jk} = 0, \text{ if } i \neq k. \]

The tensor $C_{ijk}^h$ is defined by

\[ C_{ijk}^h = \frac{1}{2} \delta_i^j \partial_k g_{jk} - \frac{1}{4} \partial_i \partial_j \partial_k F^2 \]

is known as $(h)hv$-torsion tensor [3]. It is positively homogeneous of degree -1 in directional argument and symmetric in all its indices. The $(v)hv$ - torsion tensor $C_{ijk}^h$, and its associative $(h)hv$-torsion tensor $C_{ijk}$ are related by

\[ C_{ijk}^h y^j = 0 = C_{ijk}^h y^j, \]

\[ C_{ijk}^h y^j = 0 \]

and

\[ C_{ijk}^h = g_{jk} C_{ijk}^h. \]

The $(v)hv$ - torsion tensor $C_{ijk}$ is also positively homogeneous of degree -1 in directional argument and symmetric in its lower indices.

2. Decomposition of the Curvature Tensors $R_{jkh}^+$ and $K_{jkh}^+$ in a Finsler Space Equipped with Non-Symmetric Connection

G. H. Vranceanu [8] has defined a non-symmetric connection parameter $(\Gamma_{jkh}^i \neq \Gamma_{kjh}^i)$ in n-dimensional Finsler space $F_n$

Let consider an n-dimensional Finsler space $F_n$ with non-symmetric connection parameter \( (\Gamma_{jkh}^i \neq \Gamma_{kjh}^i) \) which is based on non-symmetric fundamental tensor $g_{ij} \neq g_{ji}$.

Let write

\[ \Gamma_{jkh}^i = M_{jkh}^i + \frac{1}{2} N_{jkh}^i, \]

where $M_{jkh}^i$ and $N_{jkh}^i$ are respectively the symmetric and skew-symmetric parts of $\Gamma_{jkh}^i$. The purpose of this paper is to develop the above spaces by study the possibilities form of decompositions for the curvature tensors $R_{jkh}^+$ and $K_{jkh}^+$ which given by the conditions

\[ R_{jkh1}^+ = \lambda_i R_{jkh}^+ + \mu_i (\delta^i_h g_{jk} - \delta^i_k g_{jh}) \]

and

\[ K_{jkh1}^+ = \lambda_i K_{jkh}^+ + \mu_i (\delta^i_h g_{jk} - \delta^i_k g_{jh}) \]

respectively, where $\lambda_i$ is the $v$-covariant differential operator, and $\mu_i$ are non-zero covariant vectors.

The purpose of this paper is to develop the above spaces by study the possibilities form of decompositions for the curvature tensors $R_{jkh}^+$ and $K_{jkh}^+$.

Keywords. Generalized $R^+$ - recurrent space, Generalized $K^+$ - recurrent space, Decomposition of the curvature tensor $R_{jkh}^+$ and Decomposition of the curvature tensor $K_{jkh}^+$.

1. Introduction

C. K. Mishra and G. Lodli [4] discussed $C^h$ - recurrent and $C^v$ - recurrent Finsler space of second order and obtained different theorems regarding these spaces, also discussed the decomposability of the curvature tensor in recurrent conformal Finsler spaces. The decomposability of the curvature tensor $R_{jkh}^+$ in recurrent Finsler space equipped with non-symmetric connection parameter studied by P. Mishra, K. Srivistava and S. B. Mishra [5].

Let us consider an n-dimensional Finsler space $F_n$ equipped with the metric function $F$ satisfying the requisite condition [7].
We introduce another connection parameter $\Gamma_{k|h}^{i}\psi$ defined as order
\begin{equation}
\bar{\Gamma}_{jk}^{i} = M_{kj}^{i} - \frac{1}{2} \mathbf{N}_{kj}^{i}.
\end{equation}
With the help of the condition (2.1) and (2.2), we get
\begin{equation}
\Gamma_{jk}^{i} = \bar{\Gamma}_{jk}^{i}.
\end{equation}
Following É. Cartan [2], let a vertical stroke $\mid_j$ follow by an index denote covariant derivative with respect to $y^j$, the covariant derivative of cotravariant vector field $\bar{x}^i$ with respect to $y^j$ is defined as follows:
\begin{equation}
\bar{x}_{\mid j}^i = \partial_{j} x^i + X^i C_{ij}^j.
\end{equation}
where apostive sing below an index and following by a vertical stroke indicates that the covariant derivative has been formed with respected to the connection $\Gamma_{jk}^{i}$ as for as that index is concerned. The covariant derivative defined in (2.3) is called $\Theta -$ covariant differentiation of $X^i$ with respect to $y^j$ also is called $v-$covariant differentiation (Cartan’s covariant differentiation of the first kind).

The entities $R_{ik}^{j}$ and $K_{ik}^{j}$ are called the curvature tensor (with respect to $\Theta -$ covariant differentiation) of Finsler space with respect to non-symmetric connection parameter $\Gamma_{k|h}^{i}\psi$, such that
\begin{equation}
R_{ik|j}^{h} = \partial_{h} R_{ik}^{j} + (\partial_{j} R_{ik}^{h}) g_{k}^{h} + C_{jm}^{h} (\partial_{k} R_{jm}^{h} - G_{mh}^{k} g_{jm}) - (\partial_{j} R_{ik}^{h}) g_{k}^{h} + \Gamma_{mk}^{j} \Gamma_{mj}^{k} - \partial_{h} \Gamma_{jk}^{i} - (\partial_{k} R_{jk}^{h}) g_{h}^{k} = \Gamma_{mk}^{j} \Gamma_{mj}^{k} - \partial_{h} \Gamma_{jk}^{i} - (\partial_{k} R_{jk}^{h}) g_{h}^{k},
\end{equation}
and
\begin{equation}
K_{ik|j}^{h} = \partial_{h} K_{ik}^{j} + C_{jm}^{h} (\partial_{k} K_{jm}^{h} - G_{mh}^{k} g_{jm}) - (\partial_{j} K_{ik}^{h}) g_{k}^{h} + \Gamma_{mk}^{j} \Gamma_{mj}^{k} - \partial_{h} K_{jk}^{i} - (\partial_{k} K_{jk}^{h}) g_{h}^{k} = \Gamma_{mk}^{j} \Gamma_{mj}^{k} - \partial_{h} K_{jk}^{i} - (\partial_{k} K_{jk}^{h}) g_{h}^{k}.
\end{equation}
The curvature tensors $R_{ik|j}^{h}$ and $K_{ik|j}^{h}$ are satisfying the following:
\begin{equation}
\begin{aligned}
(2.4) \quad & a) R_{ik|j}^{h} = H_{ik}^{h} \quad b) R_{ik|j}^{h} = R_{jk}^{i}.
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
(2.5) \quad & c) K_{ik|j}^{h} = H_{ik}^{h} \quad d) K_{ik|j}^{h} = H_{jk}^{i}.
\end{aligned}
\end{equation}
Henceforth a Finsler space $F_{n}$ equipped with non symmetric connection will be written as $F_{\psi}$.

A Finsler space $F_{\psi}$ said a generalized $R^v$-non symmetric recurrent space and a generalized $K^v$- non symmetric recurrent space when the curvature tensors $R_{ik|j}^{h}$ and $K_{ik|j}^{h}$ are satisfying the following conditions
\begin{equation}
(2.5) \quad R_{ik|j}^{h} = \lambda_{i} R_{ik}^{j} + \mu_{i} (\delta_{h} g_{jk} - \delta_{k} g_{jh})
\end{equation}
and
\begin{equation}
(2.6) \quad K_{ik|j}^{h} = \lambda_{i} K_{ik}^{j} + \mu_{i} (\delta_{h} g_{jk} - \delta_{k} g_{jh}),
\end{equation}
respectively. We shall denote them briefly by $G R^v - RF_{n}$ and $K^v - RF_{n}$ respectively and the tensor which satisfies the conditions (2.5) or (2.6) will be called a generalized recurrent, where $\lambda_{i}$ and $\mu_{i}$ are non-zero covariant vectors field.

We shall discuss same of the decompositions for the curvature tensor $R_{ik|j}^{h}$ and $K_{ik|j}^{h}$ in a Finsler space equipped with non-symmetric connection for Cartan’s third and fourth curvature tensors $R_{ik|j}^{h}$ and $K_{ik|j}^{h}$ respectively.

Now, let us consider the decomposability of the curvature tensor $R_{ik|j}^{h}$ in a Finsler space $F_{\psi}$, since the curvature tensor under consideration is a mixed tensor of rank 4, hence it may be written either as a product of a vector and a tensor of rank 3 or as a product of two tensors each of rank 2.

In the first case, the possibilities forms of the decomposition of the curvature tensor $R_{ik|j}^{h}$ are as follows: $\psi$
\begin{equation}
(2.7) \quad a) R_{ik|j}^{h} = X_{i}^{l} \psi_{kj}^{h}, \quad b) R_{ik|j}^{h} = X_{j}^{l} \psi_{kl}^{i}, \quad c) R_{ik}^{h} = X_{k} \psi_{ij}, \quad d) R_{ik|j}^{h} = X_{k} \psi_{lj},
\end{equation}
In the second case, the possibilities are as follows:
\begin{equation}
(2.8) \quad a) R_{ij}^{h} = Y_{i}^{l} \psi_{jk}, \quad b) R_{ik}^{h} = Y_{k}^{l} \psi_{ij}, \quad c) R_{ik}^{h} = Y_{i}^{l} \psi_{jk}, \quad d) R_{ik|j}^{h} = Y_{l}^{j} \psi_{ik},
\end{equation}
Similarly, the possibilities form of the decomposition for the curvature tensor $K_{ik|j}^{h}$ are as follows:
\begin{equation}
(2.9) \quad a) K_{ik|j}^{h} = X_{i}^{l} \psi_{kl}^{j}, \quad b) K_{ik|j}^{h} = X_{j}^{l} \psi_{ki}, \quad c) K_{ik|j}^{h} = X_{k} \psi_{lj}, \quad d) K_{ik|j}^{h} = X_{l} \psi_{ij},
\end{equation}
In the second case, the possibilities are as follows:
\begin{equation}
(2.10) \quad a) K_{ik}^{j} = Y_{i}^{l} \psi_{lj}, \quad b) K_{ik}^{j} = Y_{j}^{l} \psi_{ki}, \quad c) K_{ik}^{j} = Y_{j}^{l} \psi_{ik}, \quad d) K_{ik}^{j} = Y_{k} \psi_{lj},
\end{equation}
Out of several possibilities given by (2.7), (2.8), (2.9) and (2.10), A. M. A. Al-Qashbari [1] and F. Y. A. Qasem and A. M. A. Al-Qashbari [6] studied the possibilities (2.7a) and (2.9a).

Our goal is to study the possibilities given by (2.7b), (2.8a), (2.9b) and (2.10a).

Let us consider a Finsler space $F_{\psi}$ whose curvature tensor $R_{ik|j}^{h}$ and $K_{ik|j}^{h}$ are decomposable in the forms (2.7b) and (2.9b), respectively. Taking the $v-$ covariant derivative for the forms (2.7b) and (2.9b), separately, with respect to $y^i$, we get
\begin{equation}
(2.11) \quad \bar{R}_{ik|j}^{h} = X_{i} \mid l \psi_{kl}^{j} + X_{j} \mid l \psi_{ij},
\end{equation}
\begin{equation}
(2.12) \quad \bar{K}_{ik|j}^{h} = X_{i} \mid l \psi_{kl}^{j} + X_{j} \mid l \psi_{ij},
\end{equation}
respectively.
Using the conditions (2.5) and (2.6) in (2.11) and (2.12), respectively, we get

\[ \lambda_i R_{jkh} + \mu_i (\delta_k^i g_{jk} - \delta_j^i g_{kj}) = X_j |^i \psi_{kh} + X_j |^i \psi_{kh}|_l \]

and

\[ \lambda_i K_{jkh} + \mu_i (\delta_k^i g_{jk} - \delta_j^i g_{kj}) = X_j |^i \psi_{kh} + X_j |^i \psi_{kh}|_l, \]

respectively.

In view of (2.7b) and (2.9b) and if the decomposable vector field \( X_j \) is covariant constant, the above equation can be written as

\[ \lambda_i X_j |^i \psi_{kh} + \mu_i (\delta_k^i g_{jk} - \delta_j^i g_{kj}) = X_j \psi_{kh} |^i \psi_{kh}|_l \]

which implies

\[ (2.13) \quad \psi_{kh}|_l = \lambda_i \psi_{kh} + \phi_i^j (\delta_k^i g_{jk} - \delta_j^i g_{kj}), \]

where \( \phi_i^j = \phi_i^j \cdot X_j \).

Thus, we conclude

**Theorem 2.1.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensor \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively, then the decomposable vector field \( \psi_{kh}^i \) is generalized recurrent, provided that the decomposable vector field \( X_j \) is covariant constant.

The condition (2.13) can be written as

\[ (2.14) \quad \psi_{kh}|_l = \lambda_i \psi_{kh} + (\eta_{ikh} - \eta_{ihkh}), \]

where \( \eta_{ikh} = \phi_i^j \delta_k^i g_{jk} \) and \( \eta_{ihkh} = \phi_i^j \delta_j^i g_{kj}. \)

Now, if the tensor field \( \eta_{ihkh} \) is skew-symmetric in the last two indices, then (2.14) can be written as

\[ (2.15) \quad \psi_{kh}|_l = \lambda_i \psi_{kh} + 2\eta_{ihkh}. \]

Thus, we conclude

**Theorem 2.2.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensor \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively and the tensor field \( \eta_{ihkh} \) is skew-symmetric in the last two indices, then the \( v \)-covariant derivative of first order for the decomposable tensor field \( \psi_{kh}^i \) is given by (2.15), provided that the decomposable vector field \( X_j \) is covariant constant.

Transvecting (2.13) by \( y_i \), using (1.9) and in view of (1.1), we get

\[ (2.16) \quad \psi_{kh}^i y_i = \lambda_i \psi_{kh} + \psi_{ihkh} + (\vartheta_{ikh} - \vartheta_{ihkh}), \]

where \( \psi_{kh} = \psi_{kh}^i y_i \), \( \psi_{ihkh} = g_{lk} \psi_{kh} \), \( \vartheta_{ikh} = \phi_i^j y_k g_{jk} \) and \( \vartheta_{ihkh} = \phi_i^j y_j g_{jk} \).

Thus, we conclude

**Theorem 2.3.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensors \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively, then the \( v \)-covariant derivative of first order for the tensor field \( \psi_{kh} \) is given by (2.16), provided that the decomposable vector field \( X_j \) is covariant constant.

The condition (2.16) can be written as

\[ (2.17) \quad \psi_{ikh} = \psi_{kh}|_l - \lambda_i \psi_{kh} + (\vartheta_{ikh} - \vartheta_{ihkh}). \]

Thus, we conclude

**Theorem 2.4.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensor \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively, then the tensor field \( \psi_{ikh} \) is defined by (2.17), provided that the decomposable vector field \( X_j \) is covariant constant.

If the tensor field \( \psi_{kh} \) is recurrent, then the formula (2.17) can be written as

\[ (2.18) \quad \psi_{ikh} = \vartheta_{ikh} - \vartheta_{ihkh}. \]

Thus, we conclude

**Theorem 2.5.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensor \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively and the tensor field \( \psi_{kh} \) is recurrent, then the tensor field \( \psi_{ikh} \) is defined by (2.18), provided that the decomposable vector field \( X_j \) is covariant constant.

If the tensor field \( \vartheta_{ihkh} \) is skew-symmetric in the last two indices, then (2.18) becomes

\[ (2.19) \quad \psi_{ikh} = 2\vartheta_{ihkh}. \]

Thus, we conclude

**Theorem 2.6.** In \( GR^R - RF_n^* \) and \( GR^V - RF_n^* \), if the curvature tensor \( R_{jkh}^i \) and \( R_{jkh}^i \) are decomposable in the forms (2.7b) and (2.9b), respectively and the tensor field \( \vartheta_{ihkh} \) is skew-symmetric in the last two indices, then the tensor field \( \psi_{ikh} \) is defined by (2.19), provided that the tensor field \( \psi_{ikh} \) is recurrent and the decomposable vector field \( X_j \) is covariant constant.

Let us consider a Finsler space \( F_n^* \) whose the curvature tensor \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.8a) and (2.10a), respectively.

Taking the \( v \)-covariant derivative for the forms (2.8a) and (2.10a), separately, with respect to \( y^i \), we get

\[ (2.20) \quad R_{jkh}^i |_{y^i} = Y^i |_{y^i} \psi_{kh} + Y^j |_{y^i} \psi_{hkh}|_l \]

and
(2.21) \[ K_{jkh}^i |_l = Y^i_j |_l \psi_{kh} + Y^i_j \psi_{kh} |_l, \]
respectively.

Using the conditions (2.5) and (2.6) in (2.20) and (2.21), respectively, we get
\[ \lambda_i R_{jkh}^i + \mu_i (\delta^j_k \delta^i g_{jkh} - \delta^i_k \delta^j g_{jkh}) = Y^i_j |_l \psi_{kh} + Y^i_j \psi_{kh} |_l \]
and
\[ \lambda_i K_{jkh}^i + \mu_i (\delta^j_k \delta^i g_{jkh} - \delta^i_k \delta^j g_{jkh}) = Y^i_j |_l \psi_{kh} + Y^i_j \psi_{kh} |_l, \]
respectively.

In view of (2.8a) and (2.10a) and if the decomposable tensor field \( Y^i_j \) is covariant constant, the above equations can be written as
\[ \lambda_i \psi_{kh} |_l = \lambda_i \psi_{kh} + \psi_{lk} (\delta^i_k \delta^j g_{jkh} - \delta^i_k \delta^j g_{jkh}), \]
which implies
\[ (2.22) \psi_{kh} |_l = \lambda_i \psi_{kh} + \psi_{lk} (\delta^i_k \delta^j g_{jkh} - \delta^i_k \delta^j g_{jkh}). \]

Thus, we conclude

**Theorem 2.7.** In \( GR^v - RF_n^v \) and \( GR^v - RE_n^v \), if the curvature tensors \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.8a) and (2.10a), respectively, then the decomposable tensor field \( \psi_{kh} \) is generalized recurrent, provided that the decomposable tensor field \( Y^i_j \) is covariant constant.

The condition (2.22) can be written as
\[ \psi_{kh} |_l = \lambda_i \psi_{kh} + \psi_{lk} (\delta^i_k \delta^j g_{jkh} - \delta^i_k \delta^j g_{jkh}), \]
where \( \psi_{lk} = \psi_{lk} \delta^i_k \delta^j g_{jkh} \) and \( \psi_{lk} = \psi_{lk} \delta^i_k \delta^j g_{jkh}. \)

Thus, we conclude

**Theorem 2.8.** In \( GR^v - RF_n^v \) and \( GR^v - RF_n^v \), if the curvature tensors \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.8a) and (2.10a), respectively, then the \( \nu \)-covariant derivative of the first order for the decomposable tensor field \( \psi_{kh} \) is given by the condition (2.23), provided that the decomposable tensor field \( Y^i_j \) is covariant constant.

Now, if the tensor field \( \psi_{lk} \) is skew-symmetric in the last two indices \( h \) and \( k \), we get
\[ \psi_{kh} |_l = \lambda_i \psi_{kh} + 2 \psi_{lk}. \]

Thus, we conclude

**Theorem 2.9.** In \( GR^v - RF_n^v \) and \( GR^v - RF_n^v \), if the curvature tensors \( R_{jkh}^i \) and \( K_{jkh}^i \) are decomposable in the forms (2.8a) and (2.10a), respectively and the tensor field \( \psi_{lk} \) is skew-symmetric in the last two indices, then the \( v \)-covariant derivative of the first order for the decomposable tensor field \( \psi_{kh} \) is given by the condition (2.24), provided that the decomposable tensor field \( Y^i_j \) is covariant constant.

### REFERENCES