Abstract: This paper presents the generalized dynamic modeling of dc motor using state space approach. A state feedback gain matrix and a minimum state observer gain matrix are designed with the help of state model of the dc motor. Both derived matrices are used to determine the transfer function of the minimum order based observer controller in MATLAB environment.

Key words: state space, minimum order state observer, pole-placement, dc motor, MATLAB.

1. Introduction

Direct current machines are the most versatile energy conversion devices. Their outstanding advantage is that the volt-ampere or speed torque characteristic of these machines are very much flexible and easily adaptable for both steady state and dynamic operations. When a wide range of speed control and torque output are required dc motor is an obvious choice [1].

The state space approach is a generalized time domain method for modeling, analyzing and designing a wide range of control systems and is particularly well suited to digital computational technique. In this paper armature current and speed of the dc motor are taken as state variables.

In this section we shall present a design method commonly called pole-placement technique. It will be shown that if the system considered is completely state controllable then poles of the closed loop system may be placed at the desired location with the help of state feedback gain matrix K.

In pole-placement approach to the design of control system we assumed that all state variables are available for feedback. In practice however all state variables are not available for feedback. So it is necessary to estimate unobservable state variables. Estimation of unmeasurable state variable is commonly called observation.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector is called reduced order state observer. If the order of the reduced order state observer is the minimum possible, the observer is called minimum order state observer [2], [3], [4].

2. DC Motor Modeling Using State Space Analysis

The different equations related to DC motor are given below

\[ e_m(t) = K_m \frac{d\theta(t)}{dt} \]  
\[ e_a(t) = L_m \frac{di_a(t)}{dt} + R_m i_a(t) + e_m(t) \]  
\[ T(t) = K_i i_a(t) \]  
\[ J \frac{d^2\theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} = T(t) \]

Where \( e_a(t) \) = armature voltage, \( e_m(t) \) = back emf, \( i_a(t) \) = armature current, \( T(t) \) = developed torque, \( \theta(t) \) = motor shaft angle, \( \frac{d\theta(t)}{dt} \) = shaft speed, J= moment of inertia of the rotor, B = viscous frictional constant, \( L_m \) = inductance of armature windings, \( R_m \) = armature winding resistance, \( K_i \) = motor torque constant, \( K_m \) = motor constant.

Here the motor speed \( \omega(t) \) is controlled by varying the armature voltage \( e_a(t) \). Hence \( e_a(t) \) is the input variable and \( \omega(t) \) is the output variable.

We chose as the state variables \( x_1(t) = \omega(t) = \frac{d\theta(t)}{dt} \) and \( x_2(t) = i_a(t) \)

The state equations will now be derived by using above equations.

\[ \frac{dx_1(t)}{dt} = \frac{B}{J} x_1(t) + \frac{K_i}{J} x_2(t) \]  
\[ \frac{dx_2(t)}{dt} = -\frac{K_m}{L_m} x_1(t) - \frac{R_m}{L_m} x_2(t) + \frac{1}{L_m} e_a(t) \]  
\[ y(t) = \frac{d\theta(t)}{dt} = \omega(t) = x_1(t) \]

Hence state model of dc motor is derived from equations (6), (7) and (8) as follows

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\frac{B}{J} & \frac{K_i}{J} \\
-\frac{K_m}{L_m} & -\frac{R_m}{L_m}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\frac{1}{L_m}
\end{bmatrix} u(t)
\]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]
3. DC Motor State Model Using Motor Parameters

Let, the motor parameters (coefficient of differential equations) are assigned to be $I_{m} = 0.5$ H. $K_{e} = 0.01$ N-m/A, $K_{p} = 0.01$ V-sec/rad, $J = 0.01$ kg-m2, $B = 0.1$ N-m-sec/rad, $R_{m} = 1 \Omega$.

Thus the state model of dc motor is derived using motor parameters and equation (9) and (10) as follows:

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
-10 & 1 \\
-0.02 & -2
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
2
\end{bmatrix} u(t)
\]

\[y(t) = [1 \ 0]
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix} \tag{11}
\]

4. State Feedback Gain Matrix K Design for DC Motor

We shall chose the control signal to be

\[u = -Kx \tag{13}\]

This means that the control signal $u$ is determined by an instantaneous state. Such a scheme is called state feedback. The $1 \times n$ matrixes $K$ is called the state feedback gain matrix.

Suppose the dc motor is given by

\[
\frac{dx}{dt} = Ax + Bu
\]

\[Y = Cx \tag{14}\]

The feedback gain matrix $K$ that forces the Eigen values of $A-KC$ to be the Eigen values of $A-BK$ can be determined by following MATLAB program.

Ackerman’s formula is used to write the program in command window. The program is given below.

\[K = \text{acker} (A,B,J)\]

\[K = 32.5 \ -4\]

5. Minimum Order State Observer Gain Matrix K Design for DC Motor

A minimum order state observer estimates fewers than $n$ state variables where $n$ is the dimension of the state vector is called minimum order state observer.

Consider equation (14) where the state vector $X$ can be partitioned into two parts $x_a$ (a scalar) and $x_b$ (a vector).

Here the state variable $x_a$ is equal to the output $y$ and thus can be directly measured and $x_b$ is the unmeasurable portion of the state vector. Then partitioned state and output equations become

\[
\begin{bmatrix}
x_a \\
x_b
\end{bmatrix} =
\begin{bmatrix}
A_{aa} & A_{ab} \\
A_{ba} & A_{bb}
\end{bmatrix}
\begin{bmatrix}
x_a \\
x_b
\end{bmatrix} +
\begin{bmatrix}
B_a \\
B_b
\end{bmatrix} u \tag{15}
\]

\[y = [1 \ 0]
\begin{bmatrix}
x_a \\
x_b
\end{bmatrix} \tag{16}\]

From equation (15) the equation for the measured portion and unmeasured portion of the state becomes.

\[
x_a = A_{aa}x_a + A_{ab}x_b + B_u \tag{17}\]

\[x_b = A_{ba}x_a + A_{bb}x_b + B_b u \tag{18}\]

Here equation (18) and (17) are known as ‘state equation’ and ‘output equation’ for the minimum order observer. The equation for the full order observer is

\[
\dot{\tilde{x}} = (A-KC)\tilde{x} + Bu + Key \tag{19}\]

Then making the substitution of table 1 into last equation we obtain

\[
\tilde{x} = (A_{bb} - k_e A_{ab}) \tilde{x}_b + A_{ba}x_a + B_b u + k_e A_{ab} x_b \tag{20}\]

<table>
<thead>
<tr>
<th>Full order state observer</th>
<th>Minimum order state observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{x}$</td>
<td>$\tilde{x}_b$</td>
</tr>
<tr>
<td>$A$</td>
<td>$A_{bb}$</td>
</tr>
<tr>
<td>$Bu$</td>
<td>$A_{ba}x_a + B_b u$</td>
</tr>
<tr>
<td>$y$</td>
<td>$x_a - A_{ab}x_b - B_ bu$</td>
</tr>
<tr>
<td>$C$</td>
<td>$A_{ab}$</td>
</tr>
<tr>
<td>$k_e$ (n×1 matrix)</td>
<td>$k_e[(n-1)\times 1 \text{ matrix}]$</td>
</tr>
</tbody>
</table>

Table 1: List of necessary substitutions for writing the observer equation for the minimum order state observer

By subtracting equation (20) from equation (18) we obtain

\[
x_b - \tilde{x}_b = (A_{bb} - k_e A_{ab}) (x_b - \tilde{x}_b) \tag{21}\]

Define $e = x_b - \tilde{x}_b$

Then equation (21) becomes

\[\dot{e} = (A_{bb} - k_e A_{ab}) e \tag{22}\]

This is the error equation for the minimum order observer. The dynamic behavior of error vector depends upon the Eigen values of $A_{bb} - k_e A_{ab}$.

Now the characteristic equation for the minimum order observer is obtained the equation as follows.
\[ SL - A_{ab} + k_e A_{ab} = (s - m_1) (s - m_2) \ldots (s - m_{n-1}) = 0 \]

Where \( m_1, m_2, \ldots, m_{n-1} \) are desired Eigen values for the minimum order observer. Suppose the desired location of the Eigen value for the minimum observer is at \( s = -9 \).

The minimum observer gain matrix is designed by the following MATLAB program. Ackerman’s formula is used to write the program in command window. The program is given below.

```matlab
>> % State observer gain matrix Ke design.
>> Aab=[1]; Abb=[-2];
>> LL=[-9];
>> Ke=acker(Abb',Aab',LL)'
Ke =
7
```

6. Response to Initial Condition with Designed K and Ke Values

The system dynamics are defined by the given equation.

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
A - BK & B k_b \\
0 & A_{ab} - k_e A_{ab}
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\quad (23)
\]

Finally, we have derived the response of the system to the given initial conditions; \( x1(0) = 1, x2(0) = 0, e1(0) = 1 \). The resulting response curves are shown in Fig. 7.

The given program is used to get the response curves for the system.

```matlab
AA=[A-B*K B*Kb;zeros(1,2) Zero Matrix];
Sys=ss(AA,eye(3),eye(3),eye(3));
t=0:0.01:8;
x0=[1;0;1];
x0=initial(sys,x0,t);
>>subplot(2,2,1);plot(t,x1);grid
>>xlabel('t(sec)');
>>ylabel('x1')
>>subplot(2,2,2);plot(t,x2);grid
>>xlabel('t(sec)');ylabel('x2')
>>subplot(2,2,3);plot(t,e);grid
>>xlabel('t(sec)');ylabel('e')
```

7. Minimum Order Observer Based Controller for DC Motor

The transfer function of minimum order observer controller is given as

\[
\frac{\bar{u}(s)}{\bar{y}(s)} = \frac{\text{num}}{\text{den}} = -[\bar{C} (SI - \bar{A})^{-1} \bar{B} + \bar{D}] \quad (24)
\]

The above transfer function can be represented by the following block diagram model.

```
where

\[
\bar{A} = \bar{A} - \hat{F} k_b
\]
\[
\bar{B} = \bar{B} - \hat{P} (k_a + k_b k_e)
\]
\[
\bar{C} = -k_b
\]
\[
\bar{D} = - (k_a + k_b k_e)
\]
\[
\hat{P} = B_b - k_e A_{ab}
\]

Fig 7: Response curves to initial condition

Fig 2: System with observed state feedback, where the observer is minimum-order observer.
The following MATLAB program is used to produce the transfer function of the minimum observer controller.

```matlab
>> A=[-10 1; 0.02 -2];
B=[0;2];C=[1 0];D=[0];
Aab=[1];Abb=[-2];
Ke=7;K=[32.5 -4];Kb=-4;
Ka=32.5;
Aaa=-10;Aba=-0.02;
Ba=0;Bb=2;
Fhat=Bb*Ke*Ba;
Ahat=Abb*Ke*Aab;
Bhat=Ahat*Ke+Ab*Kaa;
Atilde=Ahat*Fhat*Kb;
Btilde=Bhat*(Ka+Kb*Ke);
Ctilde=Kb;
Dtilde=(Ka+Kb*Ke);
>> [num,den]=ss2tf(Atilde,Btilde,Ctilde,Dtilde)

num =
4.5    12.6

den =
1     1
Hence, the transfer function of the minimum order observer controller is

\[ \frac{4.5(S+2.8)}{(S+1)} \]

The above controller is stable controller because open loop zero and open loop pole are located in the left hand side of the S plane.

8. Conclusion

The state model of dc motor has been developed with the help of different motor parameters for the proposed system. K and Ke have been determined using MATLAB. The minimum observer based controller is designed with the help of derived K and Ke values of the motor. The controller is acceptable because it is open loop as well as closed loop stable controller. Even more, the minimum order observer controller is simpler for implementation since they are dynamical system of lower order that the original systems.

REFERENCES

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Author Profile

Debabrata Pal was born in Bankura, West Bengal, India. He has received his B.Tech degree in Electrical Engineering in 2005 from West Bengal University of Technology, India and Master’s degree in Electrical Systems in 2010 from National Institute of Technology, Durgapur, India. His research interest is in Renewable Energy Sources, Control Engineering and Machine Simulation.

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