On intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) spaces

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Abstract: In this paper we have discussed some theoretical applications of intuitionistic fuzzy \( \beta \) generalized closed sets, also we have introduced intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) spaces and obtained some interesting results using this space.

Keywords: Intuitionistic fuzzy topology, intuitionistic fuzzy topological space, intuitionistic fuzzy \( \beta \) generalized closed sets, intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) space.

1. Introduction

Atanassov [1] introduces intuitionistic fuzzy sets using the notion of fuzzy sets and Coker [2] introduced intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. In this paper we have discussed some theoretical applications of intuitionistic fuzzy \( \beta \) generalized closed sets, also we have introduced intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) spaces and obtained some interesting results using this space.

2. Preliminaries

Definition 2.1: [1] An intuitionistic fuzzy set (IFS for short) \( A \) is an object having the form

\[
A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}
\]

where the functions \( \mu_A : X \to [0,1] \) and \( \nu_A : X \to [0,1] \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of non-membership (namely \( \nu_A(x) \)) of each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \) for each \( x \in X \). Denote by IFS (X), the set of all intuitionistic fuzzy sets in X.

An intuitionistic fuzzy set \( A \) in X is simply denoted by \( A = (x, \mu_A, \nu_A) \) instead of denoting \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \).

Definition 2.2: [1] Let A and B be two IFSs of the form \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \) and \( B = \{(x, \mu_B(x), \nu_B(x)) : x \in X\} \). Then,

(a) \( A \subseteq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) for all \( x \in X \),
(b) \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \),
(c) \( A^c = \{(x, \nu_A(x), \mu_A(x)) : x \in X\} \),
(d) \( A \cup B = \{(x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)) : x \in X\} \),
(e) \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x)) : x \in X\} \).

The intuitionistic fuzzy sets \( 0^\sim = (x, 0, 1) \) and \( 1^\sim = (x, 1, 0) \) are respectively the empty set and the whole set of \( X \).

Definition 2.3: [2] An intuitionistic fuzzy topology (IFT in short) on \( X \) is a family \( \tau \) of IFSs in \( X \) satisfying the following axioms:

(i) \( 0^\sim, 1^\sim \in \tau \),
(ii) \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),
(iii) \( \bigcup G_i \in \tau \) for any family \( \{G_i : i \in J\} \subseteq \tau \).

In this case the pair \( (X, \tau) \) is called an intuitionistic fuzzy topological space (IFTS in short) and any IFS in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS in short) in \( X \). The complement \( A^c \) of an IFOS \( A \) in an IFTS \( (X, \tau) \) is called an intuitionistic fuzzy closed set (IFCS in short) in \( X \).

Definition 2.4: [4] An IFS \( A = (x, \mu_A, \nu_A) \) in an IFTS \( (X, \tau) \) is said to be an

(i) intuitionistic fuzzy \( \beta \) closed set (IF\( \beta \)CS for short) if \( \text{int(cl(int(A)))} \subseteq A \),
(ii) intuitionistic fuzzy \( \beta \) open set (IF\( \beta \)OS for short) if \( A \subseteq \text{cl(cl(int(A)))} \).
Definition 2.5: [5] Let $A$ be an IFS in an IFTS $(X, \tau)$. Then the $\beta$-interior and $\beta$-closure of $A$ are defined as

$$\beta\text{int}(A) = \cup \{G / G \text{ is an IF}\beta\text{OS in } X \text{ and } G \subseteq A\},$$
$$\beta\text{cl}(A) = \cap \{K / K \text{ is an IF}\beta\text{CS in } X \text{ and } A \subseteq K\}.$$  

Note that for any IFS $A$ in $(X, \tau)$, we have $\beta\text{cl}(A^c) = (\beta\text{int}(A))^c$ and $\beta\text{int}(A^c) = (\beta\text{cl}(A))^c$.

Result 2.6: [7] Let $A$ be an IFS in $(X, \tau)$, then

(i) $\beta\text{cl}(A) \supseteq A \cup \text{int}(\text{cl}(\text{int}(A)))$

(ii) $\beta\text{int}(A) \subseteq A \cap \text{cl}(\text{int}(A))$

Definition 2.7: [7] An IFS $A$ in an IFTS $(X, \tau)$ is said to be an intuitionistic fuzzy $\beta$ generalized closed set (IF$\beta$GCS for short) if $\beta\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IF$\beta$OS in $(X, \tau)$. The complement $A^c$ of an IF$\beta$GCS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy $\beta$ generalized open set (IF$\beta$GOS in short) in $X$.

The family of all IF$\beta$GOSs of an IFTS $(X, \tau)$ is denoted by IF$\beta$GO(X).

Definition 2.8: [3] An intuitionistic fuzzy point (IFP for short), written as $p_{(\alpha, \beta)}$, is defined to be an intuitionistic fuzzy set of $X$ given by

$$p_{(\alpha, \beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0.1) & \text{otherwise}. \end{cases}$$

An intuitionistic fuzzy point $p_{(\alpha, \beta)}$ is said to belong to a set $A$ if $\alpha \leq \mu_A$ and $\beta \geq \nu_A$.

3. Applications of intuitionistic fuzzy $\beta$ generalized closed sets

The concept of intuitionistic fuzzy $\beta$ $T_{1/2}$ space was introduced by Jayanthi, D [5] in 2014. In this section we have discussed some applications of intuitionistic fuzzy $\beta$ generalized closed sets.

Definition 3.1: If every IF$\beta$GCS in $(X, \tau)$ is an IF$\beta$CS in $(X, \tau)$, then the space can be called as an intuitionistic fuzzy $\beta$ generalized $T_{1/2}$ space (IF$\beta$$_gT_{1/2}$ in short).

Example 3.2: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.4_b), (0.5_b, 0.6_a))$. Then $\tau = \{0\sim, \sim, 1\sim\}$ is an IFT on $X$. Let $A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))$ be an IFS in $X$. Then, IF$\beta$C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.

Therefore the space $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $T_{1/2}$ space, as every IF$\beta$GCS is an IF$\beta$CS in this $(X, \tau)$.

Definition 3.3: An IFTS $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $pre T_{1/2}$ (IF$\beta$$_gpreT_{1/2}$ in short) space if every IF$\beta$GCS is an IFPCS in $X$.

Example 3.4: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.5_b), (0.5_b, 0.5_a))$. Then $\tau = \{0\sim, 1\sim\}$ is an IFT on $X$. Then, IF$\beta$C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.

Now, IFPC(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \mu_a + \nu_a \leq 1 \text{ and } 0 \leq \mu_b + \nu_b \leq 1\}.

The space $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $pre T_{1/2}$ space, as every IF$\beta$GCS is an IFPCS in this $(X, \tau)$.

Definition 3.5: An IFTS $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $a T_{1/2}$ (IF$\beta$$_aT_{1/2}$ in short) space if every IF$\beta$GCS is an IFPCS in $X$.

Definition 3.6: An IFTS $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $semi T_{1/2}$ (IF$\beta$$_{semi}T_{1/2}$ in short) space if every IF$\beta$GCS is an IFSCS in $X$.

Definition 3.7: An IFTS $(X, \tau)$ is an intuitionistic fuzzy $\beta$ generalized $\alpha T_{1/2}$ (IF$\beta$$_{\alpha}T_{1/2}$ in short) space if every IF$\beta$GCS is an IFoCS in $X$.

Theorem 3.8: Every IF$\beta$$_gT_{1/2}$ space is an IF$\beta$$_{\alpha}T_{1/2}$ space but not conversely.

Proof: Let $(X, \tau)$ be an IF$\beta$$_gT_{1/2}$ space and let $A$ be an IF$\beta$GCS in $X$. By hypothesis $A$ is an IFPCS in $X$. Since every IFPCS is an IF$\beta$CS, $A$ is an IF$\beta$CS in $X$. Hence $(X, \tau)$ is an IF$\beta$$_{\alpha}T_{1/2}$ space.

Example 3.9: Let $X = \{a, b\}$ and $G = (x, (0.5_a, 0.6_b), (0.5_b, 0.4_a))$. Then $\tau = \{0\sim, 1\sim\}$ is an IFT on $X$. Now, IF$\beta$C(X) = \{0\sim, 1\sim, \mu_a \in [0,1], \mu_b \in [0,1], \nu_a \in [0,1], \nu_b \in [0,1] / \mu_b < 0.6$ whenever
\( \mu, \nu \geq 0.5, \mu \leq 0.5 \) whenever \( \mu, \nu \geq 0.6, 0 \leq \mu + \nu \leq 1 \) and

\[
\text{IF} \mathbf{\beta}(X) = \{0 \sim, 1 \sim, \mu \in [0,1], \mu \in [0,1], \nu \in [0,1], \nu \in [0,1], \mu, \nu > 0.5 \text{ whenever } \mu, \nu \leq 0.4, \mu, \nu \leq 0.5 \text{ whenever } \mu, \nu > 0.4, 0 \leq \mu, \nu \leq 1 \text{ and } 0 \leq \mu + \nu \leq 1 \}.
\]

The space \((X, \tau)\) is an intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) space, as every IF\(^{G}_{\beta}\) is an IFCS in the \((X, \tau)\), but \((X, \tau)\) is not an IF\(^{G}_{\beta}T_{1/2} \) space. Since \(A = (x, (0.5_a, 0.7_b), (0.5_a, 0.3_a))\) is an IF\(^{G}_{\beta}\) in \((X, \tau)\), but as \(cl(int(A)) = cl(G) = 1 \sim \notin A\), \(A\) is not an IFPCS.

**Theorem 3.10:** Every IF\(^{G}_{\beta}T_{1/2} \) space is an IF\(^{G}_{\beta}T_{1/2} \) space but not conversely.

**Proof:** Let \((X, \tau)\) be an IF\(^{G}_{\beta}T_{1/2} \) space and let \(A\) be an IF\(^{G}_{\beta}\) in \(X\). By hypothesis \(A\) is an IFCS in \(X\). Since every IFSCS is an IFCS, \(A\) is an IFCS in \(X\). Hence \((X, \tau)\) is an IF\(^{G}_{\beta}T_{1/2} \) space.

**Example 3.11:** Let \(X = \{a, b\}\) and \(G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b))\). Then \(\tau = \{0 \sim, G, 1 \sim\}\) is an IFT on \(X\).

Then, IF\(^{C}\)(X) = \(\{0 \sim, 1 \sim, \mu \in [0,1], \mu \in [0,1], \nu \in [0,1], \nu \in [0,1] / 0 \leq \mu, \nu \leq 1 \text{ and } 0 \leq \mu + \nu \leq 1\}.

The space \((X, \tau)\) is an intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) space, as every IF\(^{G}_{\beta}\) is an IFCS in this \((X, \tau)\), but \((X, \tau)\) is not an IF\(^{G}_{\beta}T_{1/2} \) space. Since \(A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))\) is an IF\(^{G}_{\beta}\) in \((X, \tau)\), but as \(int(cl(A)) = int(G) = G \notin A\), \(A\) is not an IFSCS.

**Theorem 3.12:** Every IF\(^{G}_{\beta}T_{1/2} \) space is an IF\(^{G}_{\beta}T_{1/2} \) space but not conversely.

**Proof:** Let \((X, \tau)\) be an IF\(^{G}_{\beta}T_{1/2} \) space and let \(A\) be an IF\(^{G}_{\beta}\) in \(X\). By hypothesis \(A\) is an IFCS in \(X\). Since every IFCS is an IFCS, \(A\) is an IFCS in \(X\). Hence \((X, \tau)\) is an IF\(^{G}_{\beta}T_{1/2} \) space.

**Example 3.13:** Let \(X = \{a, b\}\) and \(G = (x, (0.5_a, 0.4_b), (0.5_a, 0.6_b))\). Then \(\tau = \{0 \sim, G, 1 \sim\}\) is an IFT on \(X\).

Then, IF\(^{\beta}\)(X) = \(\{0 \sim, 1 \sim, \mu \in [0,1], \mu \in [0,1], \nu \in [0,1], \nu \in [0,1] / 0 \leq \mu, \nu \leq 1 \text{ and } 0 \leq \mu + \nu \leq 1\}.

The space \((X, \tau)\) is an intuitionistic fuzzy \( \beta \) generalized \( T_{1/2} \) space, as every IF\(^{G}_{\beta}\) is an IFCS in this \((X, \tau)\), but \((X, \tau)\) is not an IF\(^{G}_{\beta}T_{1/2} \) space. Since \(A = (x, (0.4_a, 0.3_b), (0.6_a, 0.7_b))\) is an IF\(^{G}_{\beta}\) in \((X, \tau)\), but as \(cl(int(A)) = cl(G) = G \notin A\), \(A\) is not an IFCS.
is an IFOS in \((X, \tau)\). Therefore \(A\) is an IFCS in \((X, \tau)\). Hence \((X, \tau)\) is an IF\(\beta_\text{g}T_{1/2}\) space.

**Theorem 3.18:** Let \((X, \tau)\) be an IF\(\beta_\text{g}T_{1/2}\) space. Then

(i) Any union of IF\(\beta GCS\) is an IF\(\beta GCS\),

(ii) Any intersection of IF\(\beta GOS\) is an IF\(\beta GOS\).

**Proof:** (i) Let \(\{A_i\}_{i \in J}\) be a collection of IF\(\beta GCS\). Since \((X, \tau)\) is an IF\(\beta_\text{g}T_{1/2}\) space, every IF\(\beta GCS\) is an IF\(\beta CS\) and hence each \(A_i\), \(i \in J\) is an IF\(\beta CS\) in \((X, \tau)\). But any union of IF\(\beta CS\) is an IF\(\beta CS\) [2], \(\cup_{i \in J} A_i\) for every \(i \in J\) is an IF\(\beta CS\). Since every IF\(\beta CS\) is an IF\(\beta GCS\) [7], \(\cup_{i \in J} A_i\) is also an IF\(\beta GCS\) in \(X\).

(ii) can be proved by taking complement in (i).

**Remark 3.19:** Not every IF\(\beta_\text{g}T_{1/2}\) space is an IFT_{1/2} space.

**Example 3.20:** Let \(X = \{a, b\}\) and \(G = \{(x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\}\). Then \(\tau = \{0\sim, 1\sim, G\}\) is an IFT\(_X\) on \(X\).

Then, IF\(\beta C(X) = \{0\sim, 1\sim, \mu_x \in [0,1], \nu_x \in [0,1], \nu_b \in [0,1] / 0 \leq \mu_x + \nu_x \leq 1\text{ and } 0 \leq \mu_b + \nu_b \leq 1\}\).

Since all IF\(\beta GCS\) in \(X\) are IF\(\beta CS\) in \((X, \tau)\), this is an IF\(\beta_\text{g}T_{1/2}\) space. But it is not an IFT\_{1/2} space since if \(A = \{(x, (0.6_a, 0.7_b), (0.4_a, 0.3_b)\}\}), then \(\text{cl}(A) = 1\sim \not\subseteq 1\sim\) whenever \(A \not\subseteq 1\sim\) and therefore \(A\) is an IF\(\beta CS\) in \(X\) but as \(\text{cl}(A) = 1\sim \not\subseteq A\), \(A\) is not an IF\(\beta CS\) in \(X\). Therefore \((X, \tau)\) is not an IFT\_{1/2} space.

**Theorem 3.21:** For any IFS \(A\) in \((X, \tau)\) where \(X\) is an IF\(\beta_\text{g}T_{1/2}\) space, \(A \in IF\(\beta GOS(X)\) if and only if for every IFP \(p_{(a,b)} \in A\), there exists an IF\(\beta GOS\) in \(X\) such that \(p_{(a,b)} \in B \subseteq A\).

**Proof:** Necessity: If \(A \in IF\(\beta GOS(X)\), then we can take \(B = \emptyset\) so that \(p_{(a,b)} \in B \subseteq A\) for every IFP \(p_{(a,b)} \in A\).

Sufficiency: Let \(A\) be an IFS in \((X, \tau)\) and assume that there exists \(B \in IF\(\beta GOS(X)\) such that \(p_{(a,b)} \in B \subseteq A\). Since \(X\) is an IF\(\beta_\text{g}T_{1/2}\) space, \(B\) is an IF\(\beta OS\). Then \(A = \bigcup_{p_{(a,b)} \in B} \{p_{(a,b)}\} \subseteq \bigcup_{p_{(a,b)} \in B} \{p_{(a,b)}\} \subseteq A\). Therefore \(A = \bigcup_{p_{(a,b)} \in B} \{p_{(a,b)}\} \subseteq A\), which is an IF\(\beta OS\). Hence \(A\) is an IF\(\beta GOS\).

**Theorem 3.22:** Let \((X, \tau)\) be an IF\(\beta_\text{g}T_{1/2}\) space, then the following conditions are equivalent:

(i) \(A \in IF\(\beta GOS(X)\)

(ii) \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\)

(iii) \(\text{cl}(A) \in IF\(\beta RC(X)\).

**Proof:** (i) \(\Rightarrow\) (ii) Let \(A\) be an IF\(\beta GOS\). Then since \(X\) is an IF\(\beta_\text{g}T_{1/2}\) space, \(A\) is an IF\(\beta OS\). Therefore \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\).

(ii) \(\Rightarrow\) (iii) Let \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\). Then \(\text{cl}(A) \subseteq \text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(A)) \subseteq \text{cl}(A)\). Therefore \(\text{cl}(A) = \text{cl}(\text{int}(A))\). Hence \(\text{cl}(A) \in IF\(\beta RC(X)\).

(iii) \(\Rightarrow\) (i) Since \(\text{cl}(A)\) is an IFRCS, \(\text{cl}(A) = \text{cl}(\text{int}(\text{cl}(A)))\) and since \(A \subseteq \text{cl}(A)\), \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\). Therefore \(A\) is an IF\(\beta OS\). Hence by [8], \(A \in IF\(\beta GOS(X)\).

**Theorem 3.23:** Let \((X, \tau)\) be an IFT\_{1/2} and let \(X\) be an IF\(\beta_\text{g}T_{1/2}\) space, then the following conditions are equivalent:

(i) \(A \in IF\(\beta GC(X)\)

(ii) \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\)

(iii) \(\text{int}(A) \in IF\(\beta RO(X)\).

**Proof:** (i) \(\Rightarrow\) (ii) Let \(A\) be an IF\(\beta GCS\). Then since \(X\) is an IF\(\beta_\text{g}T_{1/2}\) space, \(A\) is an IF\(\beta CS\). Therefore \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\).

(ii) \(\Rightarrow\) (iii) Let \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\). Then \(\text{int}(A) \supseteq \text{int}(\text{cl}(\text{int}(A))) = \text{int}(\text{int}(A)) \supseteq \text{int}(\text{int}(A)) = \text{int}(A)\). Therefore \(\text{int}(\text{int}(A)) = \text{int}(A)\). Hence \(\text{int}(A) \in IF\(\beta RO(X)\).

(iii) \(\Rightarrow\) (i) Since \(\text{int}(A)\) is an IFROS, \(\text{int}(A) = \text{int}(\text{cl}(\text{int}(A)))\) and since \(\text{int}(A) \subseteq \text{cl}(A)\), \(\text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(A)\). Therefore \(A\) is an IF\(\beta CS\) which implies \(A^c\) is an IF\(\beta OS\). Hence by [8], \(A^c \in IF\(\beta GOS\). Therefore \(A \in IF\(\beta GC(X)\).

4. **References**


